Verhulst Function for Modeling Minerals Depletion and Product Manufacture

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Minerals Depletion

The Verhulst Function is a good function to use for fitting minerals-depletion data. It is used to model the fact that their is usually an exponential growth in the rate of extraction a mineral from the Earth, followed by a peak, after which the extraction rate declines exponentially. The function is

$$P(t) = \frac{Q_\infty}{n\tau} \frac{(2^n - 1)\exp\left(\frac{t - t_{\frac{1}{2}}}{\tau}\right)}{\left[1 + (2^n - 1)\exp\left(\frac{t - t_{\frac{1}{2}}}{\tau}\right)\right]^{\frac{n+1}{n}}}$$ ..

Q_∞ is the amount to be eventually extracted, τ is the rising exponential time constant, $n\tau$ is the falling exponential time constant and $t_{1/2}$ is the time at which the resource is one-half depleted. The parameter n determines the amount of skewing at large times. For n = 1 the extraction curve is symmetrical and the peak occurs at $t_{1/2}$. The deviation of the peak time from $t_{1/2}$ is negative for n > 1 (skewed toward large times) and is positive for n < 1 (skewed toward small times).

The maximum of P(t) occurs at $t_{\max} = t_{\frac{1}{2}} + \tau \ln\left(\frac{n}{2^n - 1}\right)$, which yields $P_{\max}(t_{\max}) = \frac{Q_\infty}{\tau} \frac{1}{(n+1)^{\frac{n+1}{n}}}$. Note that, for the symmetric case (n=1): $t_{\max} = t_{1/2}$ and $P_{\max}(t_{\max}) = Q_\infty / 4\tau$.

When a peak is symmetrical (n = 1), the Verhulst function simplifies to

$$P(t) = \frac{Q_\infty}{\tau} \frac{\exp\left(\frac{t - t_{1/2}}{\tau}\right)}{\left[1 + \exp\left(\frac{t - t_{1/2}}{\tau}\right)\right]^2}.$$

The asymmetry parameter, n, must be greater than 0. For the case of n = 0, the Verhulst function becomes the Gompertz function:

$$P(t) = \ln 2 \frac{Q_\infty}{\tau} \exp\left(\frac{t - t_{1/2}}{\tau}\right)\left(\frac{1}{2}\right)^{\exp\left(\frac{t - t_{1/2}}{\tau}\right)}.$$

Amount Left to be Extracted

The amount left to be extracted at time t is $Q(t) = \frac{Q_\infty}{\left[1 + \left(2^n - 1\right)\exp\left(\frac{t - t_{1/2}}{\tau}\right)\right]^{1/n}}$.

The amount already extracted is $Q_X(t) = Q_\infty - \frac{Q_\infty}{\left[1 + \left(2^n - 1\right)\exp\left(\frac{t - t_{1/2}}{\tau}\right)\right]^{1/n}}$.

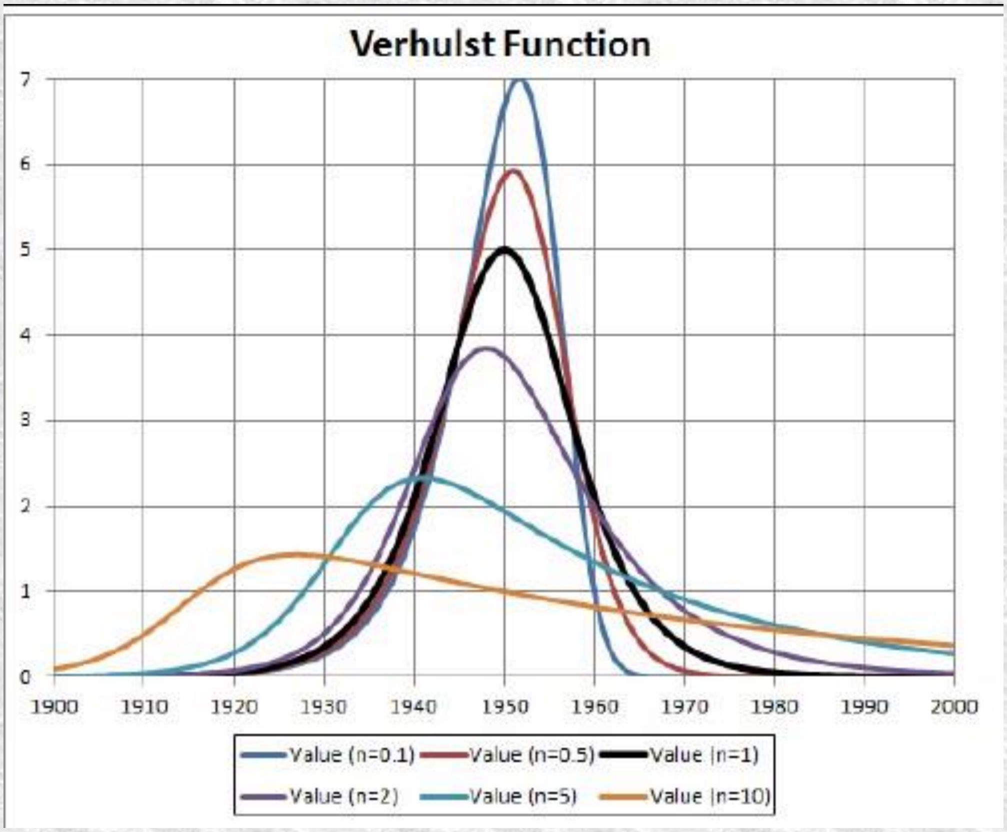
This function is related to the hyperbolic-tangent function when n = 1 and the two Q_∞ 's are different. 1 & 2:

$$\tanh\left(\frac{t}{2}\right) = 1 - \frac{2}{1 + \exp(t)};\ \text{proof:} 1 - \frac{2}{1 + \exp(t)} = \frac{\exp(t) - 1}{1 + \exp(t)} * \frac{\exp\left(-\frac{t}{2}\right)}{\exp\left(-\frac{t}{2}\right)} = \frac{\exp\left(\frac{t}{2}\right) - \exp\left(-\frac{t}{2}\right)}{\exp\left(\frac{t}{2}\right) + \exp\left(-\frac{t}{2}\right)} = \tanh\left(\frac{t}{2}\right).$$

Therefore, one can use $V(t) = 1 - \frac{2}{\left[1 + \left(2^n - 1\right)\exp(t)\right]^{1/n}}$ as a generalized $\tanh\left(\frac{t}{2}\right)$ that allows for asymmetry.

Graphs

The following graph shows the Verhulst function for Q_∞ = 100, $t_{1/2}$ = 1950 and τ = 5 with 6 different values of n:



The area under all the curves is Q_∞ = 100.

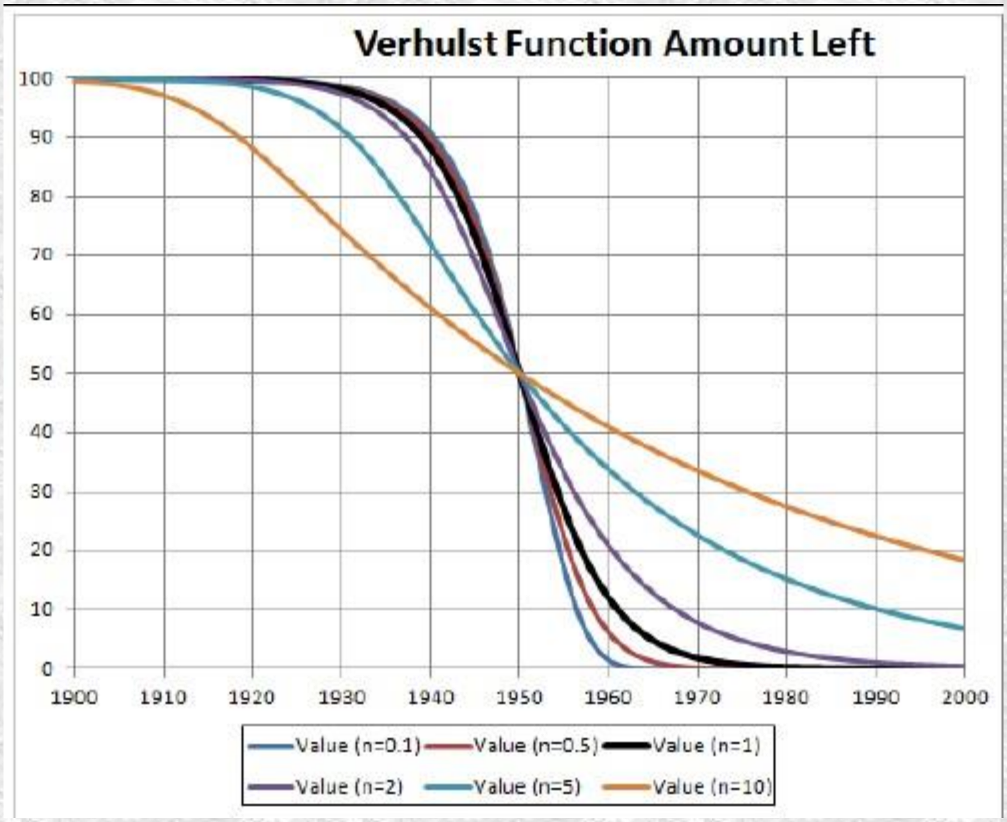
For a symmetric Verhulst function used to fit data up to a time t and given the definitions

A = amount already extracted and R = true reserves = amount left to be extracted:

$$R = \frac{A+R}{1+\exp\left(\frac{t-t_{1/2}}{\tau}\right)} \text{ or } A = R\exp\left(\frac{t-t_{1/2}}{\tau}\right). \text{ Thus, } \frac{t-t_{1/2}}{\tau} = \ln\left(\frac{A}{R}\right) \text{ or } \boxed{t_{1/2} = t - \tau\ln\left(\frac{A}{R}\right)}.$$

For a known A, t and τ, the peak position, $t_{1/2}$, varies logarithmically with 1/R, which is much less than linearly. If enough data are present in the extraction exponential rise, the rate, τ, can be determined by fitting those data by the Verhulst function. A can be determined by adding up the extraction for all years up to year t.

The following graph shows the amount-left Verhulst function for Q_∞ = 100, $t_{1/2}$ = 1950 and τ = 5 with 6 different values of n:



Of course, the amount already extracted at time t is Q_∞ - Q(t).

Duration of Extraction

It is useful to define a "duration" for the extraction of a mineral by the difference in the times when (f-1)/f of it has been extracted and when 1/f of it has been extracted:

$$D = \tau \ln\left[\frac{(f^n - 1)(f-1)^n}{f^n - (f-1)^n}\right].$$

This is derived from

$$Q(t_-) = \frac{Q_\infty}{\left[1+\left(2^n-1\right)\exp\left\{\frac{t_- - t_{1/2}}{\tau}\right\}\right]^{1/n}} = \frac{Q_\infty}{f}; \quad Q(t_+) = \frac{Q_\infty}{\left[1+\left(2^n-1\right)\exp\left\{\frac{t_+ - t_{1/2}}{\tau}\right\}\right]^{1/n}} = \frac{(f-1)Q_\infty}{f}$$

and solving for $D = t_+ - t_-$.

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For the symmetric case (n = 1) $D = \tau\left[\ln(f-1) - \ln\left(\frac{1}{f-1}\right)\right] = 2\tau\ln(f-1)$.

A good choice for f is 10; then the duration would be the time interval for extracting the middle 80% of Q_∞ .

Fast Decline

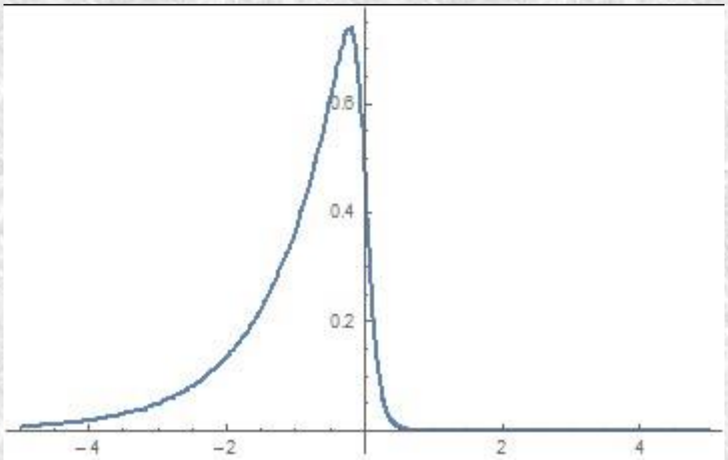
The Verhulst function has a limit as to how fast extraction can decline. See the curve for n = 0.1 above; putting in parameter n much closer to zero does not change that curve very much. However, the Verhulst function appears to work quite well for minerals extraction; steeper extraction appears to be unlikely. Of course, common bankruptcies of all companies extracting a rare mineral could cause a very steep extraction decline.

Another exponential function that allows much faster decline is:

$$C\frac{\exp\left(\frac{t-t_0}{\tau}\right)}{1+\exp\left(\frac{t-t_0}{n\tau}\right)} \text{ for } 0 < n < 1.$$

This gives a peaked function only for n < 1.

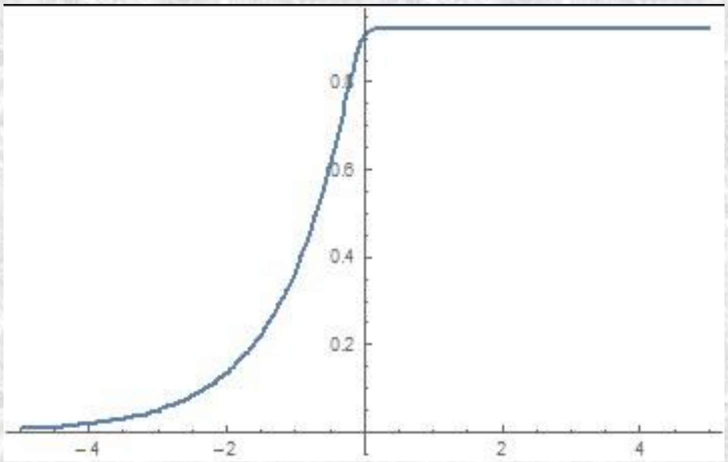
Here is an example curve for n = 0.09 (The declining exponential time constant 0.09 times smaller than the rising time constant.):



Values of n closer to 0 yield a very steep decline, approaching a zero time constant.

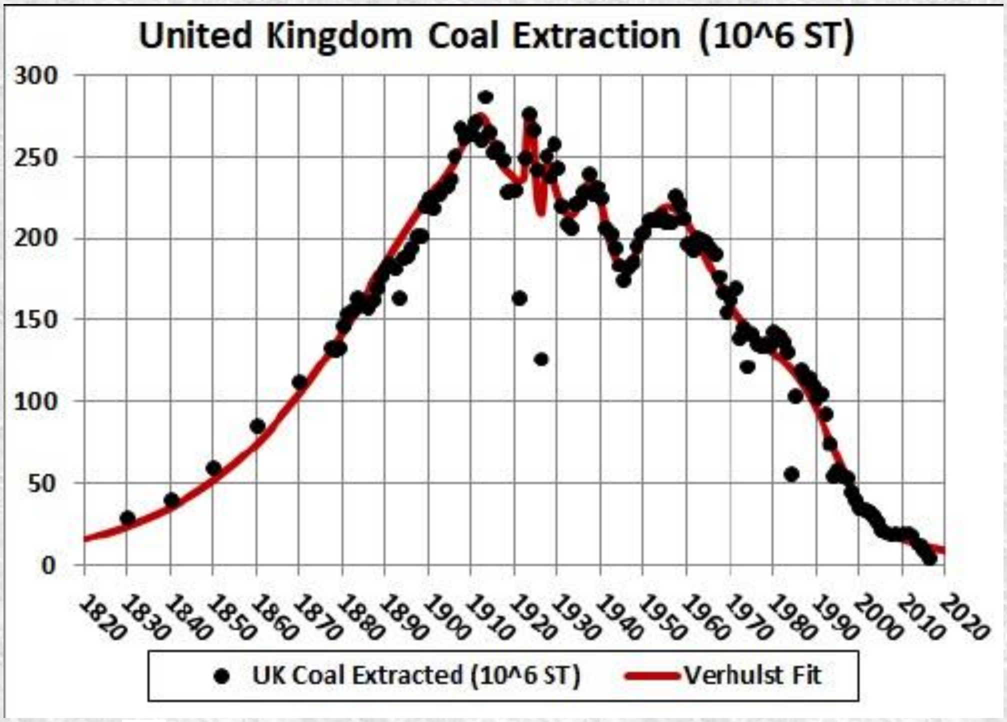
To get the cumulative extraction for a mineral, one needs the integration of the extraction function. Unfortunately, the integration of this Fast-Deline function yields the complicated Hypergeometric function. Math programs, such as Mathematica, Excel and Scientific Workplace have this function.

Here is an example cumulative curve for n = 0.09:



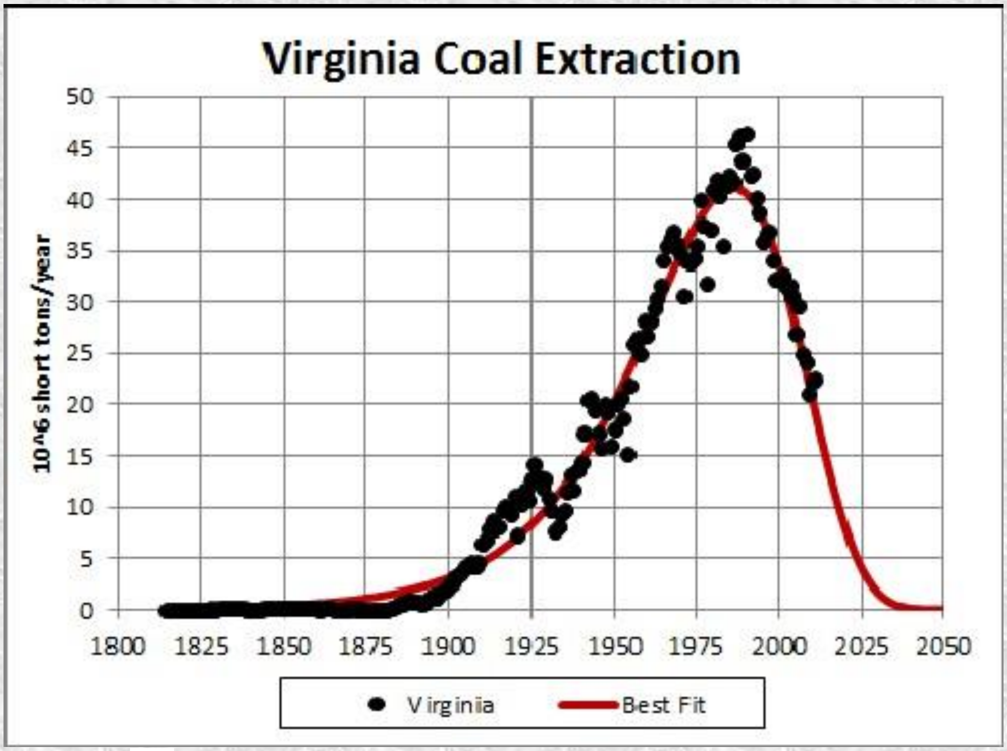
Examples

Coal Extraction in United Kingdom



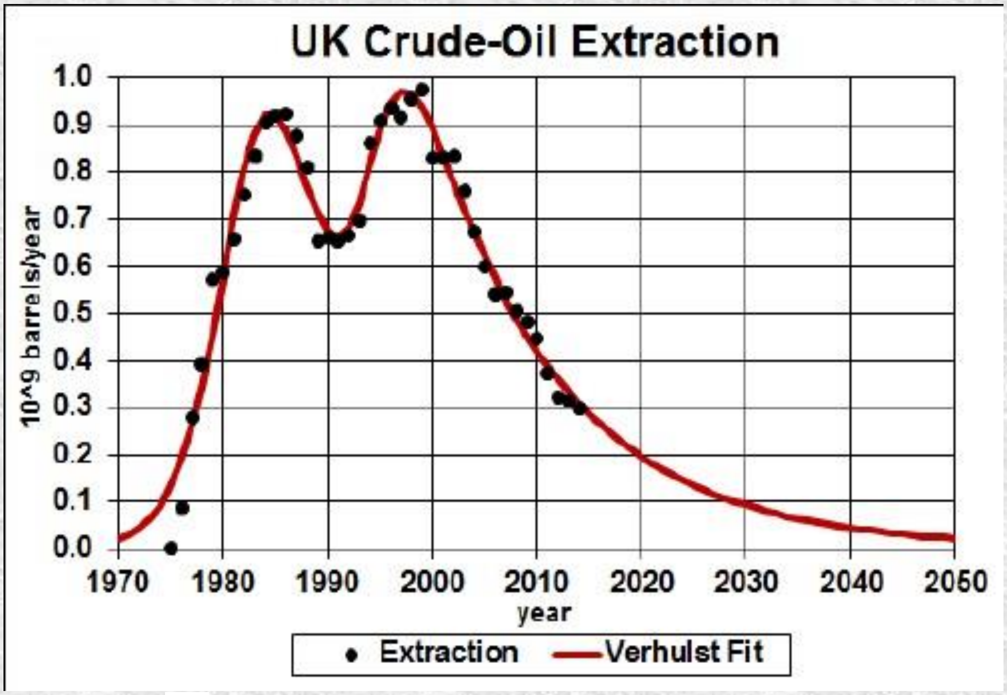
Q_∞ =32422 x 10^6 tons, $t_{1/2}$ = 1913.5, τ = 47.50 and n = 0.1 .

Coal Extraction in Virginia USA



Q_∞ =2720 x 10^6 tons, $t_{1/2}$ = 1977.3, τ = 24.22 and n = 0.1 .

Crude-Oil Extraction in United Kingdom



First peak: Q_∞ =13.52 x 10^9 barrels, $t_{1/2}$ = 1985.9, τ = 2.645 and n = 2.316 .

Second peak: Q_∞ =17.07 x 10^6 tons, $t_{1/2}$ = 2004.6, τ = 1.876 and n = 7.565 .

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Product Manufacture

When a new product is introduced in a factory it often starts out at a low level, then later rises to new levels at time goes on. One could represent this moving from one level to another by a series of hyperbolic tangents transiting from one level to an asymptotic next level.. However, the rising and the asymptotic exponential time constants are the same for a hyperbolic tangent. The Verhulst function can have different rising and the asymptotic exponential time constants (τ and $n\tau$).

The equation for moving from one manufacturing level, Q_L, to another, Q_H, is

$$Q_M(t) = Q_H - \frac{Q_H - Q_L}{\left[1 + \left(2^n - 1\right)\exp\left(\frac{t - t_{1/2}}{\tau}\right)\right]^{1/n}}.$$

The cumulative production is just the summation of the periodic production data.

If there are many levels the production equation is

$$Q_M(t) \equiv Q_{N_{\max}} - \sum_{N=1}^{N_{\max}} \frac{Q_N - Q_{N-1}}{\left[1+\left(2^{n_N}-1\right)\exp\left(\frac{t-t_N}{\tau_N}\right)\right]^{1/n_N}}$$

, starting at N=0 to $N=N_{max}$.

If the periodic production data are highly uncertain it would be easier to use the hyperbolic-tangent function to fit the levels' data. The rising and the asymptotic exponential time constants are the same for the hyperbolic tangent.

References

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- http://roperld.com/science/MultiStatesMathematics.pdf
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