Gompertz Function from Verhulst Function

The Verhulst Function (http://www.roperld.com/science/minerals/VerhulstFunction.htm) is

$$\boxed{p_v(x) = \frac{Q}{nw}\frac{(2^n-1)\exp\left(\frac{x-x_0}{w}\right)}{\left[1+(2^n-1)\exp\left(\frac{x-x_0}{w}\right)\right]^{\frac{n+1}{n}}}}.$$

To simplify further calculations, set $Q = 1$, $w = 1$ and $x_0 = 0$:

$$\boxed{p_v(x) = \frac{(2^n-1)\exp x}{n[1+(2^n-1)\exp(x)]^{\frac{n+1}{n}}}}.$$

The Gompertz Function is

$$\boxed{p_g(x) = \ln 2\frac{Q}{w}\exp\left(\frac{x-x_0}{w}\right)\left(\frac{1}{2}\right)^{\exp\left(\frac{x-x_0}{w}\right)}}.$$

To simplify further calculations, set $Q = 1$, $w = 1$ and $x_0 = 0$:

$$\boxed{p_g(x) = \ln 2\exp(x)\left(\frac{1}{2}\right)^{\exp x}}.$$

Show that

$$\boxed{p_g(x) = \lim_{n\to 0} p_v(x)}.$$

That is

$$\ln 2\left(\frac{1}{2}\right)^{\exp x} = \lim_{n\to 0}\frac{2^n-1}{n[1+(2^n-1)\exp(x)]^{\frac{n+1}{n}}}$$

Take the $\ln$ of both sides:

$$\ln\left[\ln 2\left(\frac{1}{2}\right)^{\exp(x)}\right] = \ln(\ln 2) + \exp(x)\ln\left(\frac{1}{2}\right) = \boxed{\ln(\ln 2) - e^x\ln 2}$$

$$\ln\left(\frac{2^n-1}{n[1+(2^n-1)\exp(x)]^{\frac{n+1}{n}}}\right) = \ln(2^n-1) - \ln n - \frac{n+1}{n}\ln[1+(2^n-1)\exp(x)]$$

Expand 2^n in powers of n: $\boxed{2^n = 1 + n\ln 2 + O(n^2)}$

Then

$$\ln\left(\frac{2^n-1}{n[1+(2^n-1)\exp(x)]^{\frac{n+1}{n}}}\right) =$$

$$\ln(1+n\ln 2-1) - \ln n - \frac{n+1}{n}\ln[1+(1+n\ln 2-1)\exp(x)] =$$

$$\ln(n\ln 2) - \ln n - \frac{1}{n}(n+1)\ln(ne^x\ln 2+1) =$$

$$\ln n + \ln(\ln 2) - \ln n - \frac{1}{n}(n+1)\ln(ne^x\ln 2+1) = \qquad \boxed{\ln(\ln 2) - \frac{1}{n}(n+1)\ln(ne^x\ln 2+1)}$$

Show that $\boxed{e^x\ln 2 = \lim_{n\to 0}\frac{1}{n}(n+1)\ln(ne^x\ln 2+1)}$

Expand in powers of n: $\boxed{\ln(ne^x\ln 2+1) = ne^x\ln 2 + O(n^2)}$

$$\lim_{n\to 0}\frac{1}{n}(n+1)ne^x\ln 2 = e^x\ln 2$$

QED!