Depletion Theory

L. David Roper (roperld@vt.edu) 1976, revised 2000

Introduction

An outline of a theory of depletion of nonrenewable resources is presented here. Start with two basic irrefutable facts:

1. The earth (or any portion of the earth) is a finite source of any mineral.
2. As a mineral is extracted from the earth it becomes steadily more difficult to extract the remainder. By "more difficult" is meant that more materials and energy are required and more environmental degradation occurs.

These two facts define complicated nonlinear interactions among all minerals: The increasing scarcity with time of one mineral (say crude oil) makes it more difficult to obtain another mineral (say iron ore) which may be crucially important in the extraction of the first mineral. This is only one of scores of nonlinear interactions that are simultaneously at work.

Common sense tells one the kind of long-term "average" production-rate behavior to expect for any mineral. There are components of both technology and sociology that interplay in the behavior:

1. In the *earliest times* the mineral is relatively readily available, but the technology for its extraction and society's need for it are undeveloped. Therefore, the production rate will increase slowly at first. However, as the extracted mineral enters into the mainstream of the society its presence will generate more need for it and thereby generate more advanced extraction technology. Thus, it is reasonable to assume that the production rate at earliest times will be some increasing function of the amount already extracted at that time. Let $Q_\infty \equiv$ amount that will be eventually extracted and $Q(t) \equiv$ amount left to be extracted at time t; then the production rate $P(t) = -dQ/dt$ at time t is some function of $[Q_\infty - Q(t)]$. Since any smooth function can be expanded as a power series in an independent variable, at the very earliest times $P(t)$ should be proportional to some power of $[Q_\infty - Q(t)]$. The simplest assumption and one that often works in other similar situations is that $P(t) \propto [Q_\infty - Q(t)]$ at the earliest times. However, more complicated possibilities will be considered.
2. At the *latest times* when the mineral is almost completely depleted, the principal limitation on the production rate P will be the amount left to be extracted $Q(t)$ at that time. Again, at the very latest times P should be proportional to some power of $Q(t)$. The simplest assumption and one that often works in other similar situations is that $P(t) \propto Q(t)$ at the latest times. However, more complicated possibilities will be considered.
3. 3. At *intermediate times* there are no rational arguments that we can muster for any particular functional form for $P(t)$ as a function of $Q(t)$. So we shall consider several possibilities and let the production data for a given mineral "choose" which of the possibilities works best by performing least-squares fits to the data. Some obvious statements can be made, however: After rising slowly at earliest times, the production rate should begin to accelerate, then later (at an inflection point) decelerate until the production rate peaks at some time. Then the rate will begin to decline in a similar, but not necessarily symmetrical, fashion. Finally, $P(t)$ will asymptotically approach zero. The simplest assumption that one could make which yields this kind of behavior is that $P(t)$ is strictly proportional to the first power of both $[Q_\infty - Q(t)]$ and $Q(t)$ at all times; i.e.

$$P(t) = -\frac{dQ}{dt} = kQ(Q_\infty - Q)/Q_\infty \qquad \text{Eq. (1)}$$

where k is a rate constant that is a measure of the usefulness of the mineral and the long-term economic conditions of the society. (One can define a time constant $\tau = 1/k$.) This equation, as well as more complicated cases, are discussed below.

Of course, in reality for a particular mineral the long-term average behavior described above will not precisely describe the production-rate behavior. There are short-term social phenomena, such as wars and economic depressions, that can and sometimes do cause rather large fluctuations in the production rates. (A detailed study of correlations of these mineral-production fluctuations with specific social phenomena would be interesting. We do not attempt it here.) These short-term fluctuations exhibit behavior similar to that described above for the long-term average behavior except that the rate constant k is greatly increased (time constant t is greatly reduced). (We shall often refer to the long-term average behavior as the "background" behavior.) There are two situations that could exist:

1. The short-term fluctuations have little or no effect on the long-term background behavior. That is, the rate constant for the background behavior is unchanged as short-term fluctuations occur.
2. The short-term fluctuations are evidence of changes in the long-term use of that mineral either because of the onset of new long-term social phenomena or new mineral technology (e.g., substitution of another mineral for it in its major use). That is, the rate constant for the background behavior is changed as short-term fluctuations occur.

Of course, it is possible that the long-term background rate "constant" k is not really a constant in time even in the absence of fluctuations. In fact, one would think that after a mineral has become an integral part of a society's *modus operandi* that the society will make a large effort to keep its production rate up when it otherwise would decline sharply for constant k. That is, the society's increased efforts to extract the mineral will cause k to decrease with time rather than be constant. This will cause the production-rate curve to be asymmetrically skewed toward large times; most nearly depleted United States minerals have such skewed production-rate curves. Also, gradual substitution of one mineral for another (e.g., oil for coal) could cause k to change with time.

Logistic Equation

Eq. (1) given above for the production rate $P(t)$ as a function of the amount $Q(t)$ yet to be extracted at time t can be solved to yield the widely-used logistic equation:

$$Q(t) = \frac{1}{2} Q_\infty \left[1 - \tanh\left(\frac{t - t_{1/2}}{2\tau} \right) \right], \qquad \text{Eq. (2)}$$

where $\tau = 1/k$ is the time constant and $t_{1/2}$ is the time at which the mineral is one-half depleted. Label $t_{1/2}$ as the "half date". Then

$$P(t) = \frac{Q_\infty}{4\tau} \left[1 - \tanh^2\left(\frac{t - t_{1/2}}{2\tau} \right) \right]$$

Variable Decay-Rate Model

One can complicate the simple model developed above by assuming that the decay rate, $k(t)$, is a function of time. Then

$$P(t) = -\frac{dQ(t)}{dt} = k(t) \frac{Q(t)}{Q_\infty} [Q_\infty - Q(t)], \qquad \text{Eq. (3)}$$

which has the solution

$$Q(t) = \frac{1}{2} Q_\infty \left[1 - \tanh\left(\frac{g(t) - g(t_{1/2})}{2\tau} \right) \right], \qquad \text{Eq. (4)}$$

where

$$g(t) \equiv \int^t k(t)dt \, .$$

This approach could be used to give the large-time skewing that often occurs in nearly depleted production data.

Generalized Verhulst Equation

A more complicated, but still analytically solvable, case that contains Eqs. (2) and (4) above as special cases is the Verhulst equation, defined by:

$$P(t) = -\frac{dQ(t)}{dt} = \frac{k(t)Q(t)}{n}\left[1 - \left\{\frac{Q(t)}{Q_\infty}\right\}^n\right], \qquad \text{Eq. (5)}$$

where $n \equiv$ "asymmetry parameter". It has the solution

$$Q(t) = \frac{Q_\infty}{[1 + (2^n - 1)\exp\{g(t) - g(t_{1/2})\}]^{1/n}} \qquad \text{Eq. (6)}$$

$$\xrightarrow{\tau = 1/k = \text{constant}} \frac{Q_\infty}{\left[1 + (2^n - 1)\exp\left(\frac{t - t_{1/2}}{\tau}\right)\right]^{1/n}} \, .$$

Then

$$P(t) = \frac{Q_\infty}{n\tau}\frac{(2^n - 1)\exp\left(\frac{t - t_{1/2}}{\tau}\right)}{\left[1 + (2^n - 1)\exp\left(\frac{t - t_{1/2}}{\tau}\right)\right]^{1/n}} \, .$$

Note that this appoach assumes that $P(t)$ is linear in $Q(t)$ for large times but is nonlinear in $[Q_\infty - Q(t)]$ for small times.

For $n = 1$ Eq. (6) can be shown to be the same as Eq. (4), which is the same as Eq. (2) when $k(t)$ is constant in t. However, even with $k =$ constant, Eq. (6) contains a large-time skewing if $n > 1$. (The generalized Verhulst curve is skewed toward short times for $0 \geq n > 1$, is symmetrical for $n = 1$ and is skewed toward large times for $n > 1$.) We choose Eq. (6) in fitting the nearly-depleted skewed data rather than the Eq. (4) because it is mathematically simpler.

Gompertz Equation

Another special case of the Verhulst equation [Eq. (6)] is when $k =$constant and $n = 0$. This yields the Gompertz equation, which is given by the equations:

$$P(t) = -\frac{dQ(t)}{dt} = \frac{1}{\tau}Q(t)\ln\left[\frac{Q(t)}{Q_\infty}\right] \qquad \text{Eq. (7)}$$

and

$$Q(t) = Q_\infty\left(\frac{1}{2}\right)^{\exp[(t - t_{1/2})/\tau]} \, . \qquad \text{Eq. (8)}$$

This curve is skewed toward short times. Then

$$P(t) = \ln 2\frac{Q_\infty}{\tau}\exp\left(\frac{t - t_{1/2}}{\tau}\right)\left(\frac{1}{2}\right)^{\exp\left[\left(t - t_{1/2}\right)/\tau\right]} \, .$$

Error Function Equation

There are at least two other symmetric peak functions that often occur in nature. One is the Gaussian function, which is given by the equations:

$$P(t) = -\frac{dQ(t)}{dt} = \frac{Q_\infty}{\tau\sqrt{\pi}} \exp\left[-\left(\frac{t - t_{1/2}}{\tau} \right)^2 \right] \qquad \text{Eq. (9)}$$

and

$$Q(t) = \frac{1}{2} Q_\infty \left[1 - \operatorname{erf}\left(\frac{t - t_{1/2}}{\tau} \right) \right]. \qquad \text{Eq. (10)}$$

Lorentzian Equation

Another symmetric peak function that often occurs in nature is the Lorentzian function, which is given by the equations:

$$P(t) = -\frac{dQ(t)}{dt} = \frac{Q_\infty}{\pi\tau} \left[\frac{1}{1 + \left(\frac{t - t_{1/2}}{\tau} \right)^2} \right] \qquad \text{Eq. (11)}$$

and

$$Q(t) = \frac{1}{\pi} Q_\infty \cot^{-1}\left(\frac{t - t_{1/2}}{\tau} \right). \qquad \text{Eq. (12)}$$

One can show that this is a situation in which $P(t)$ is proportional to $[Q_\infty - Q(t)]^2$ for short times and to $Q^2(t)$ for long times, but is **not** proportional to $Q^2(t)[Q_\infty - Q(t)]^2$ for intermediate times.

Appendix: Verhulst Equation

The solution to $-\frac{dQ(t)}{dt} = \frac{k(t)Q(t)}{n}\left[1 - \left\{ \frac{Q(t)}{Q_\infty} \right\}^n \right]$ can be found by induction by solving the ODE for different values of n and then inducing for abitrary n: $Q(t) = \frac{(-1)^{1/n} a}{\left(e^{kt} - a^n \right)^{1/n}}$, where a is an integration constant which must be evaluated by a boundary condition, which is difficult. I used the "boundary condition" that I know the solution should be Eq. 6 above. That is, I assume that result and check to see if it agrees with the result given above and what the constant a is to get that result. I get the constant a to be

$$a = -\frac{1}{(2^n - 1)^{1/n}}.$$