Pion-Nucleon Scattering Equations

http://arxiv.org/PS_cache/arxiv/pdf/0909/0909.4390v2.pdf
http://teachers.web.cern.ch/teachers/archiv/HST2002/Bubblech/mbitu/applications_of_special
http://en.wikipedia.org/wiki/Four-momentum

Kinematics

$N(p) + \pi(q) \rightarrow N(p') + \pi(q')$, where $p^2 = \vec{p}^{\,2} + M^2$ and $q^2 = \vec{q}^{\,2} + m^2$ where $(c = 1)$.
Masses:
$M_i =$ initial nucleon; $M_f =$ final nucleon; $m_i =$ inital pion; $m_f =$ final pion.
$m_{+-} = 139.570$ MeV, $m_0 = 134.977$ MeV, $M_p = 938.272 = 6.7212 m_{+-}$, $M_n = 939.566$.
Center of Momentum System: $\vec{p} + \vec{q} = \vec{p'} + \vec{q'} = 0$. Thus, $\vec{p} \cdot \vec{p'} = |\vec{p}|\left|\vec{p'}\right| \cos\theta_{cm}$

http://skisickness.com/2010/04/25/:
Four-momentum squared is invariant:
Lab system:

$$s = (w + W)^2 - \left(\vec{p} + \vec{q}\right) \cdot \left(\vec{p} + \vec{q}\right) = (m + T + M)^2 - \left[(m + T)^2 - M^2\right]$$

$$\boxed{s = (m + M)^2 + 2MT}$$

CM system:

$$s = (w' + W')^2 = \boxed{\left(\sqrt{k^2 + m^2} + \sqrt{k^2 + M^2}\right)^2 = s}$$ because $\vec{p'} = -\vec{P'} = \vec{k}$.

$$k = \sqrt{\frac{\left(s - m^2 - M^2\right)^2 - 4m^2M^2}{4s}} = \sqrt{\frac{\left[(m+M)^2 + 2MT - m^2 - M^2\right]^2 - 4m^2M^2}{4\left[(m+M)^2 + 2MT\right]}} = \sqrt{\frac{4M^2T(T+2m)}{4\left[(m+M)^2 + 2MT\right]}}$$

$$\boxed{k = M\sqrt{\frac{T(T+2m)}{(m+M)^2 + 2MT}}}$$

$$\left[(m + M)^2 + 2MT - m^2 - M^2\right]^2 - 4m^2M^2 = 4TM^2(T + 2m)$$

$$(m + T + M)^2 - \left[(m + T)^2 - M^2\right]$$

$$\boxed{s = W^2 = \left(\sqrt{k^2 + m^2} + \sqrt{k^2 + M^2}\right)^2}$$, Solution is:

$$k = \frac{1}{2s}\sqrt{s^3 + M^4s + m^4s - 2M^2s^2 - 2m^2s^2 - 2M^2m^2s} = \sqrt{\frac{s^2 + M^4 + m^4 - 2M^2s - 2m^2s - 2M^2m^2}{4s}}$$

$$\boxed{k = \sqrt{\frac{\left(s - m^2 - M^2\right)^2 - 4m^2M^2}{4s}}}$$

because

$$(s - m^2 - M^2)^2 - 4m^2M^2 = M^4 + m^4 + s^2 - 2M^2s - 2m^2s - 2M^2m^2$$

Pion Momentum and Energy as Functions of the Total Energy W

$$k = \sqrt{\frac{\left(s - m^2 - M^2\right)^2 - 4m^2M^2}{4s}} = \sqrt{\frac{\left(W^2 - m^2 - M^2\right)^2 - 4m^2M^2}{4W^2}}$$

or $$\boxed{k = \frac{1}{2W}\sqrt{\left[W^2 - (M + m)^2\right]\left[W^2 - (M - m)^2\right]} = \frac{1}{2W}\sqrt{W^4 - 2(M^2 + m^2)W^2 + (M^2 - m^2)^2}}$$

because

$$\left[W^2 - (M + m)^2\right]\left[W^2 - (M - m)^2\right] = W^4 - W^2\left[(M + m)^2 + (M - m)^2\right] + (M^2 - m^2)^2$$

$= W^4 - 2W^2(M^2 + m^2) + (M^2 - m^2)^2$

$(W^2 - m^2 - M^2)^2 - 4m^2M^2 = M^4 + m^4 + W^4 - 2M^2m^2 - 2M^2W^2 - 2m^2W^2$

and

$W^4 - 2(M^2 + m^2)W^2 + (M^2 - m^2)^2 = M^4 + m^4 + W^4 - 2M^2m^2 - 2M^2W^2 - 2m^2W^2.$

$q = \sqrt{m^2 + k^2} = \sqrt{m^2 + \frac{(W^2 - m^2 - M^2)^2 - 4m^2M^2}{4W^2}} = \frac{1}{2W}\sqrt{4m^2W^2 + (W^2 - m^2 - M^2)^2 - 4m^2M^2}$

or $\boxed{q = \frac{1}{2W}\sqrt{W^4 - 2(M^2 - m^2)W^2 + (M^2 - m^2)^2} = \frac{1}{2W}[W^2 - (M^2 - m^2)]}$ because

$4m^2W^2 + (W^2 - m^2 - M^2)^2 - 4m^2M^2 = M^4 + m^4 + W^4 - 2M^2m^2 - 2M^2W^2 + 2m^2W^2$

and

$W^4 - 2(M^2 - m^2)W^2 + (M^2 - m^2)^2 = M^4 + m^4 + W^4 - 2M^2m^2 - 2M^2W^2 + 2m^2W^2.$

$\therefore \boxed{p = W - q = \frac{1}{2W}[W^2 + (M^2 - m^2)]}$

Threshold

Threshold lab kinetic energy: $E_0 = \frac{\left(\sum_i M_i\right)^2 - (M_i + M_T)}{2M_T}$.

Threshold c.m. momentum: $k_0 = M_T\sqrt{\frac{E_0(E_0 + 2M_i)}{(M_T + M_i)^2 + 2M_TE_0}}$.

$$\pi^- + p \to 2\pi^0 + n : E_0 = 1.14998 = 160.503\ MeV; k_0 = 1.47644$$

$$\pi^\pm + p \to \pi^0 + \pi^\pm + p : E_0 = 1.180505 = 164.7633\ MeV; k_0 = 1.49905$$

The mass m' of a particle produced along with a nucleon of mass M:

$$\boxed{m' = -M + \sqrt{(M+1)^2 - 2(M - k_0^2) + 2\sqrt{(M - k_0^2)^2 + k_0^2(M+1)^2}}}$$

in units of the incident-particle mass.

Resonances

$A = -\frac{\Gamma_e}{2(W - W_r) + i\Gamma_t} = -\frac{2\Gamma_e(W - W_r)}{4(W - W_r)^2 + \Gamma_t^2} + i\frac{\Gamma_e\Gamma_t}{4(W - W_r)^2 + \Gamma_t^2}$, where $\boxed{\Gamma_t = \Gamma_e + \Gamma_i}$.

$\boxed{\Gamma_e = k^{2\ell+1}\Gamma_{e0}}$ and $\boxed{\Gamma_i = (k - k_i)^{2\ell+1}\Gamma_{i0} \text{ for } k \geqslant k_{in}}$.

$\boxed{\mathrm{Re}A = -\frac{2\Gamma_e(W - W_r)}{4(W - W_r)^2 + \Gamma_t^2}}$ and $\boxed{\mathrm{Im}A = \frac{\Gamma_e\Gamma_t}{4(W - W_r)^2 + \Gamma_t^2}}$

More generally for the P_{11} partial wave: $\boxed{\Gamma_{el} = \frac{4M(W - W_z)}{W(W + W_r)}\gamma_{el}\frac{(kr)^3}{1 + (kr)^2}}$ and

$\boxed{\Gamma_{in} = \gamma_{in}\frac{[r_{in}(k - k_{in})]^3}{1 + [r_{in}(k - k_{in})]^2} \text{ for } k \geqslant k_{in}}$

Basic Equations

Total Cross Section

$\sigma_T = \frac{4\pi\bar{\lambda}}{k}\mathrm{Im}f(0)$ where, for $\pi^\pm$:

$\bar{\lambda} = \frac{\hbar}{\mu c} = \frac{6.58211899\times10^{-16}}{\frac{139.57018}{c^2}c}\frac{eV\text{–}s}{MeV}m/s = \frac{6.58211899\times10^{-16}(2.99792458\times10^8)}{139.57018}\times 10^{-3}\frac{MeV\text{–}s}{MeV}m/s = 1.4138\times10^{-12}$

$\frac{6.58211899\times10^{-16}(2.99792458\times10^8)}{134.9766} = 1.4619\times10^{-9}\ m$ for π^0.

Differential Cross Section for Unpolarized Target

$\sigma(\theta) = |f(\theta)|^2 + |g(\theta)|^2$ where θ is the c.m. pion scattering angle.

Recoil-Nucleon Polarization for Unpolarized Target

$\boldsymbol{P}(\theta) = -\frac{2\,\mathrm{Im}[f^*(\theta)g(\theta)]}{\sigma(\theta)}\hat{n}$ where $\hat{n} = \frac{\boldsymbol{k}\times\boldsymbol{k}'}{|\boldsymbol{k}\times\boldsymbol{k}'|}$, k incident pion c.m. momentum and $\boldsymbol{k}'$ final pion c.m. momentum.

Differential Cross Section for Polarized Target

$\sigma(\theta) = |f(\theta)|^2 + |g(\theta)|^2 - 2\,\mathrm{Im}[f^*(\theta)g(\theta)](\hat{n}\cdot\boldsymbol{P}_i)$ where $\boldsymbol{P}_i$ is the initial target polarization.

Recoil-Nucleon Polarization for Polarized Target

$$\boldsymbol{P}_f(\theta) = -\frac{2\,\mathrm{Im}[f^*(\theta)g(\theta)]\hat{n} - 2\,\mathrm{Re}[f^*(\theta)g(\theta)](\hat{n}\times\boldsymbol{P}_i) + \left(|f(\theta)|^2 + |g(\theta)|^2\right)(\hat{n}\cdot\boldsymbol{P}_i)\hat{n} - \left(|f(\theta)|^2 - |g(\theta)|^2\right)[\hat{n}\times(\hat{n}\times\boldsymbol{P}_i)]}{\sigma(\theta)}.$$

Nonspin-Flip Amplitude

$f(\theta) = \frac{\bar{\lambda}}{k}\sum_{\ell=0}^{\ell_m}[(\ell+1)A_{\ell+} + \ell A_{\ell-}]P_\ell(\cos\theta)$ where ℓ_m is the maximum value of the angular momentum ℓ used in the analysis.

Spin-Flip Amplitude

$$g(\theta) = \frac{\bar{\lambda}}{k}\sum_{\ell=0}^{\ell_m}[A_{\ell+} - A_{\ell-}]P_\ell^1(\cos\theta)$$

Partial-Wave Amplitude

$A_{\ell\pm} = \frac{1}{2i}[\eta_{\ell\pm}\exp(2i\delta_{\ell\pm}) - 1]$ where $\delta_{\ell\pm}$ is the phase shift and $\eta_{\ell\pm}$ is the absorption parameter for scattering in the states with total angular momentum $j = \ell \pm 1$.

Therefore,

$$\mathrm{Re}A = \tfrac{1}{2}\eta\sin 2\delta$$

$$\mathrm{Im}A = \tfrac{1}{2}(1 - \eta\cos 2\delta)$$

$$\tan 2\delta = \frac{2\,\mathrm{Re}A}{1 - 2\,\mathrm{Im}A}$$

$$\eta^2 = (2\,\mathrm{Re}A)^2 + (1 - 2\,\mathrm{Im}A)^2$$

Observable Amplitudes in Terms of Isotopic-Spin Amplitudes

$A^+_{\ell\pm} = A^{(3)}_{\ell\pm}$, $A^+_{\ell\pm} = \frac{1}{3}\left(A^{(3)}_{\ell\pm} + 2A^{(1)}_{\ell\pm}\right)$ and $A^{cx}_{\ell\pm} = \frac{\sqrt{2}}{3}\left(A^{(3)}_{\ell\pm} - A^{(1)}_{\ell\pm}\right)$ where the superscripts have the following meanings:

$$+ : \quad \pi^+ + p \to \pi^+ + p$$

$$- : \quad \pi^- + p \to \pi^- + p$$

$$cx : \quad \pi^- + p \to \pi^0 + n$$

$$(3) : \quad T = \tfrac{3}{2}$$

$$(1) \quad T = \tfrac{1}{2}$$