General Relativity Mathematics

L. David Roper, roperld@vt.edu

Introduction

- A contravariant vector is one which transforms like $v^{\mu} = \frac{dx^{\mu}}{d\tau}$ where x^{μ} are the coordinates of a particle at its proper time τ. $\boxed{x^{\mu} = (ct, x, y, z) = \text{contravariant spacetime}}$.
- A covariant vector is one which transforms like $\Phi_{\mu} = \frac{d\Phi}{dx^{\mu}}$, where Φ is a scalar field. Note the placement of the index being upper for a contravariant vector and being lower for a covariant vector. $\boxed{x_{\mu} = (-ct, x, y, z) = \text{covariant spacetime}}$. $\boxed{x_{\mu} = \eta_{\mu\nu} x^{\mu}}$, where
$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{metric tensor or Minkowski tensor.}$$
- A repeated index implies summation; e.g., $\boxed{g_{\mu\nu} x^{\mu} \equiv \sum_{\nu=0}^{4} g_{\mu\nu} x^{\mu}}$ and $\boxed{\frac{\partial y^{\nu}}{\partial x_{\mu}} \equiv \partial_{\mu} y^{\nu} \equiv y^{\nu}_{,\mu}}$..
- The tensors described below are of rank 2 because they are related to the spacetime vector, $r_{\mu} = (-ct, x, y, z)$, a tensor of rank 1. A rank-2 tensor can be represented by a 4x4 matrix.
- **Coordinate time** = time between two events as measured by an observer's clock.
- **Spacetime Metric Equation =** $\boxed{ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} = \eta_{\mu\nu} dx'^{\mu} dx'^{\nu}}$ is invariant for all inertial reference frames (IRF); it is equivalent to the Pythagorean Theorem in plane geometry.
    - **Spacelike** if $ds^2>0$; **Lightlike** if $ds^2=0$; **Timelike** if $ds^2<0$.
- **Proper time** = time as measured on a time-like world line by a clock moving along that line. $d\tau = \sqrt{-ds^2} = \boxed{dt\sqrt{1-v^2} = d\tau}$.
- $A \cdot B = \eta_{\mu\nu} A^{\mu} A^{\mu}$ and A=A $\cdot$ A.
- **Kronecker Delta:** $\delta^{\mu}{}_{\nu} = \begin{bmatrix} 0 \text{ if } \mu = \nu \\ 1 \text{ if } \mu \neq \nu \end{bmatrix}$.
- **Coordinate Basis: u, v, w & ; e_u is tangent to w curve increasing u; e_w is tangent to w curve increasing w.**
    - $e_u \cdot e_w$ may be nonzero; $\mathbf{e_u}$ may not have unit length; $\mathbf{e_u}$ may change in magnitude or direction.
    - $A = A^{\mu} e_{\mu}$; $ds = dx^{\mu} e_{\mu}$; $\boxed{ds^2 = dx^{\mu} dx^{\nu} e_{\mu} e_{\nu} \equiv g_{\mu\nu} dx^{\mu} dx^{\nu}}$; $g_{\mu\nu} = \text{metric tensor.}$
    - $dx'^{\mu} = \partial_{\nu} x'^{\mu} dx^{\nu}$; $\boxed{A'^{\mu} = \partial_{\nu} x'^{\mu} A^{\nu}}$; $A^{\mu} = \partial_{'\nu} x^{\mu} A'^{\nu}$; $\boxed{\partial_{\nu} x'^{\mu} \partial_{'\nu} x^{\mu} = \delta^{\mu}{}_{\nu}}$.
    - $ds^2 = g'_{\mu\nu} dx'^{\mu} dx'^{\nu} = g_{\mu\nu} dx^{\mu} dx^{\nu}$; $\boxed{g'_{\mu\nu} = \partial_{'\mu} x^{\alpha} \partial_{'\nu} x^{\beta} g_{\alpha\beta}}$ and $g_{\mu\nu} = \partial_{\mu} x'^{\alpha} \partial_{\nu} x'^{\beta} g'_{\alpha\beta}$.

- **Inverse metric tensor**: $\boxed{g^{\mu\alpha} g_{\alpha\nu} = g'^{\mu\alpha} g'_{\alpha\nu} = \delta^{\mu}{}_{\nu}}$; $\boxed{g'^{\mu\nu} = \partial_\alpha x'^{\mu} \partial_\beta x'^{\nu} g^{\alpha\beta}}$.

Tensors

Rank n	# Components	Other Name	Symbol	Transformation Law	Example Quantities
0	$4^0 = 1$	(invariant scalar)	Φ	$\Phi' = \Phi$	Rest energy m
1	$4^1 = 4$	vector covector	A^μ A_μ	$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$ $A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu$	four-velocity u^μ four-momentum p^μ gradient $\partial_\mu \Phi$ of a scalar
2	$4^2 = 16$	tensor	$T^{\mu\nu}$ $T^\mu{}_\nu$ $T_{\mu\nu}$	$T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}$ $T'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} T^\alpha{}_\beta$ $T'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta}$	Inverse metric $g^{\mu\nu}$ EM field tensor $F^{\mu\nu}$ Stress-energy tensor $T^{\mu\nu}$ Kronecker delta $\delta^\mu{}_\nu$ Metric tensor $g_{\mu\nu}$ Ricci tensor $R_{\mu\nu}$ Einstein tensor $G_{\mu\nu}$
3	$4^3 = 64$	tensor	$M^{\mu\nu}{}_\alpha$	$M'^{\mu\nu}{}_\alpha = \frac{\partial x'^\mu}{\partial x^\beta} \frac{\partial x'^\nu}{\partial x^\gamma} \frac{\partial x^\sigma}{\partial x'^\alpha} M^{\beta\gamma}{}_\sigma$	(no obvious examples)
4	$4^4 = 256$	tensor	$R^\alpha{}_{\beta\mu\nu}$	$R'^\alpha{}_{\beta\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x^\delta}{\partial x'^\beta} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} R^\gamma{}_{\delta\sigma\lambda}$	Riemann tensor $R^\alpha{}_{\beta\mu\nu}$

$$\boxed{T'^{\mu\nu\ldots}{}_{\sigma\rho\ldots} = \partial_\beta x'^\mu \partial_\gamma x'^\nu \ldots \partial'_\sigma x^\tau \partial'_\rho x^\kappa \ldots T^{\beta\gamma\ldots}{}_{\tau\kappa\ldots}}$$

Operation	Example	Result	Comments
tensor sum	$p_1^\mu + p_2^\mu = p_{\text{tot}}^\mu$	tensor of same rank	Addition is defined only between tensors having the same rank and index positions, and whose corresponding components have the same units.
tensor product	$A^\mu B_\nu = M^\mu{}_\nu$	tensor of rank $n_1 + n_2$	The components of the resulting tensor are the products of all possible pairings of the input tensor components. The values of n_1 and n_2 are the ranks of the input tensors
contraction over an upper and lower index	$R^\alpha{}_{\mu\alpha\nu} = R_{\mu\nu}$	tensor of rank $n - 2$	Starting tensor must have a rank ≥ 2, and the sum must be over an upper and lower index
raising an index	$g^{\mu\alpha} R_{\alpha\nu} = R^\mu{}_\nu$	tensor of same rank	
lowering an index	$g_{\mu\alpha} T^{\alpha\nu} = T_\mu{}^\nu$	tensor of same rank	

1. An index that is repeated twice within the same term, and where one instance is upper and the other lower, is to be summed over.
2. A repeated index must not occur more than twice within a single term.
3. Any index that is not repeated must occur in the same position (up or down) in all terms in an equation. (Exception: it's allowed to set a tensor expression to zero.)
4. Any repeated (otherwise known as 'dummy' or 'bound') index may be renamed to any other symbol, provided it doesn't violate any of the other rules.
5. Any single (otherwise known as 'free') index may be renamed to any other symbol, provided that symbol also occurs once only in each term.

Natural Units in General Relativity

$$c = 3\times10^{8}\text{ms}^{-1} = 1 = c \Rightarrow \boxed{\text{s} = 3\times10^{8}\text{m}},\ \boxed{a = \text{m}/\text{s}^2 = 0.111\times10^{-16}\text{m}^{-1}}$$

$$\boxed{c = G \equiv 1}: 1 = \frac{G}{c^2} = \frac{6.674\times10^{-11}\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}}{3\times10^{8}\text{m}^2\text{s}^{-2}} = \boxed{0.742\times10^{-27}\text{kg}^{-1}\,\text{m} = 1}$$

So, convert the units of any kinematic variable using $\boxed{\text{s} = 3\times10^{8}\,\text{m}}$ and $\boxed{\text{kg} = 0.742\times10^{-27}\,\text{m}}$

$$p = \text{kg}\,\text{ms}^{-1} = \frac{0.742\times10^{-27}}{3\times10^{8}}\text{m} = 2.4733\times10^{-36}\text{m}$$

$$L = \text{kg}\,\text{m}^2\,\text{s}^{-1} = \frac{0.742\times10^{-27}}{3\times10^{8}}\text{m} = 2.4733\times10^{-36}\text{m}$$

Special Relativity

$$\boxed{\begin{array}{cc} t' = \gamma(t-\beta x) & t = \gamma(t'+\beta x') \\ x' = \gamma(-\beta t + x) & x = \gamma(\beta t' + x') \\ y' = y & y = y' \\ z' = z & z = z' \end{array}}$$

where $\boxed{\beta = \frac{V}{c} \text{ and } \gamma = \sqrt{\frac{1}{1-\beta^2}}}$.

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \quad \text{where } \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

$$\begin{bmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

Rotation in Newtonian space x-y plane: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Rotation in SR (t,x) space: $\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}$, where $\cosh\theta = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma$ & $\sinh\theta = \beta\gamma$.

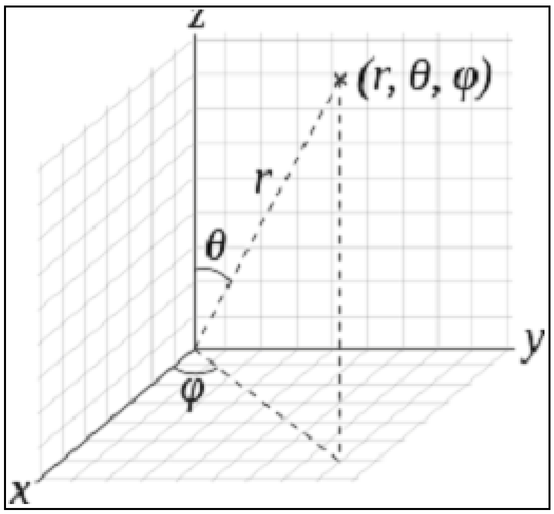
Note that $\left[\begin{array}{l} -dt'^2 + dx'^2 = -\gamma^2(dt-\beta dx)^2 + \gamma^2(-\beta dt + dx)^2 = \gamma^2\left(\beta^2 dt^2 - \beta^2 dx^2 + dx^2 - dt^2 - 2\beta dt dx + 2\beta dt dx\right) = \\ = \frac{1}{1-\beta^2}\left[-(1-\beta^2)dt^2 + (1-\beta^2)dx^2\right] = -dt^2 + dx^2;\ \text{i.e., } ds^2 = -dt^2 + dx^2 \text{ is invariant.} \end{array}\right.$

[Velocities transform](#) as:
$$\begin{cases} v_x' = \dfrac{dx'}{dt'} = \dfrac{\gamma\left(-\beta dt + dx\right)}{\gamma\left(dt - \beta dx\right)} = \dfrac{-\beta + dx/dt}{1-\beta\left(dx/dt\right)} = \boxed{\dfrac{v_x - \beta}{1-\beta v_x} = v_x'}, \\ v_y' = \dfrac{dy'}{dt'} = \dfrac{dy}{\gamma\left(dt - \beta dx\right)} = \dfrac{dy/dt}{\gamma\left(1-\beta\left(dx/dt\right)\right)} = \boxed{\dfrac{v_y\sqrt{1-\beta^2}}{1-\beta v_x} = v_y'} \text{ and} \\ v_z' = \dfrac{dz'}{dt'} = \dfrac{dz}{\gamma\left(dt - \beta dx\right)} = \dfrac{dz/dt}{\gamma\left(1-\beta\left(dx/dt\right)\right)} = \boxed{\dfrac{v_z\sqrt{1-\beta^2}}{1-\beta v_x} = v_z'}. \end{cases}$$

For a constant force, see https://xphysics.wordpress.com/2010/11/07/relativistic-acceleration-due-to-a-constant-force/.

For other forces, see http://www.reed.edu/physics/courses/Physics411/html/page2/files/Lecture.11.pdf.

[Spherical Coordinates](#)

[

$$\boxed{x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta}.$$

Unit Vectors

$$\boxed{\begin{gathered} \hat{r} = \sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z} \\ \hat{\theta} = \cos\theta\cos\varphi + \cos\theta\sin\varphi\hat{y} - \sin\theta\hat{z} \\ \hat{\varphi} = -\sin\varphi\hat{x} + \cos\varphi\hat{y} \end{gathered}}.$$

[Line Element](#)

$$\boxed{c^2 d\tau^2 = -c^2 dt^2 + dr^2 + d\theta^2 + d\varphi^2}.$$

Parabolic Coordinates

$p(x,y) = x;\ q(x,y) = y - cx^2,\ c = \text{constant}.$

$$x(p,q) = p;\ y(p,q) = cp^2 + q.$$

Four Vectors

4-Displacement: $ds = [dt, dx, dy, dx]$

4-velocity: $\boxed{u = ds/d\tau} = [dt/d\tau, dx/d\tau, dy/d\tau, dz/d\tau] = \boxed{[u^t, u^x, u^y, u^z] = u}$.

Transformation to a new IRF: $\begin{bmatrix} u'^t \\ u'^x \\ u'^y \\ u'^z \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u^t \\ u^x \\ u^y \\ u^z \end{bmatrix}$.

Scalar Product of 4-vectors

$$A \cdot B = -A^t B^t + A^x B^x + A^y B^y + A^z B^z \;\&\; A^2 = A \cdot A = -A^t A^t + A^x A^x + A^y A^y + A^z A^z.$$

$$ds \cdot ds = ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

3-Velocity

Since $\boxed{d\tau = dt' = \sqrt{-ds^2} = dt\sqrt{1-v^2} = dt/\gamma}$, $\boxed{\therefore u = [u^t, u^x, u^y, u^z] = \gamma[1, v_x, v_y, v_z]}$ & $\boxed{u \cdot u = -1}$ $\boxed{\text{0-mass } u \cdot u = 0}$.

$v = [v_x, v_y, v_z] = [u^x, u^y, u^z]/u^t$; For $v \ll 1: u = [1, v_x, v_y, v_z]$, for $v = 0: u = [1, 0, 0, 0]$.

$$\boxed{u_x = \frac{u'_x + v}{1 + vu'_x}, u_y = \frac{\sqrt{1-v^2}u'_y}{1 + vu'_x}, u_z = \frac{\sqrt{1-v^2}u'_z}{1 + vu'_x}}.$$

4-momentum

$$p = mu = [p^t, p^x, p^y, p^z] = \gamma m[1, v_x, v_y, v_z] \;\&\; \boxed{p \cdot p = m^2 u \cdot u = -m^2} \boxed{\text{0 mass: } p \cdot p = 0}.$$

For $v \ll 1$: $p^x \approx mv_x$, $p^y \approx mv_y$, $p^z \approx mv_z$. 4-momentum is conserved.

Relativistic Energy and Momentum

Relativistic energy: $\boxed{E = p^t = \gamma m}$; Relativistic momentum: $\boxed{\vec{p} = (p\ , p_y, p_z) = \gamma m\vec{v}}$; $\boxed{E^2 = \vec{p}^2 + m^2}$; $\boxed{\vec{v} = \vec{p}/E}$.

For $v \ll 1: E = m + mv^2/2 + \ldots; \gamma = 1 + v^2/2 + 3v^4/8 + \ldots$

Define kinetic energy: $E = m + K = \gamma m \rightarrow \boxed{K = m(\gamma - 1)}$.

4-momentum of Light

$p = E[1, v_x, v_y, v_z]$, as well as for all objects. Photon rest mass = $m = \sqrt{E^2 - \vec{p}^2} = E\sqrt{1-v^2} = 0$ since $v = 1$.

Energy in Observer's Frame:

$$-p \cdot u_{obs} = -(-p^t \cdot 1 + \vec{p} \cdot 0) = p^t = E.$$

Since the scalar product is IRF independent, it can be calculated in any IRF.

Einstein General Relativity Equation

$$\boxed{G^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \equiv \kappa T^{\mu\nu}}$$, where

$\boxed{G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}} =$ Einstein tensor, $\Lambda =$ cosmological constant, $g^{\mu\nu} =$ metric tensor,

$G =$ Newton gravitational constant, $T_{\mu\nu} =$ stress-energy tensor, $\boxed{R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = R_{\nu\mu}}$ = Ricci curvature tensor,

$\boxed{R^{\mu\nu} = g^{\mu\beta} g^{\nu\sigma} R_{\beta\sigma},\ R = g^{\mu\nu} R_{\mu\nu} = \text{Ricci curvature scalar}}$,

$\boxed{R^{\rho}_{\mu\lambda\nu} = \partial_{\lambda}\Gamma^{\rho}_{\nu\mu} - \partial_{\nu}\Gamma^{\rho}_{\lambda\mu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} = \text{Riemann curvature tensor}}$,

$R_{\rho\mu\lambda\nu} = g_{\nu\rho} R^{\rho}_{\mu\lambda\nu},\quad R^{\nu}_{\mu;\nu} = \frac{1}{2}\partial_{\mu} R,$

where $\boxed{\Gamma^{\rho}_{\nu\mu} = \text{Christoffel symbol} = \frac{1}{2} g^{\rho\lambda}\left(\partial_{\mu} g_{\lambda\nu} + \partial_{\nu} g_{\lambda\mu} - \partial_{\lambda} g_{\nu\mu}\right)}$.

Riemann Curvature Tensor Properties

$$\boxed{\begin{array}{c} R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha} = -R_{\nu\mu\alpha\beta},\quad R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}. \\ R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} = 0 = \text{First Bianchi Identity} \equiv R_{\mu[\nu\alpha\beta]}. \\ R_{\mu\nu\alpha\beta;\gamma} + R_{\mu\nu\gamma\alpha;\beta} + R_{\mu\nu\beta\gamma;\alpha} = 0 = \text{Second Bianchi Identity} \equiv R_{\mu\nu[\alpha\beta;\gamma]} \end{array}}.$$

Conservation of four-momentum: $\boxed{\nabla_{\mu} T^{\mu\nu}} = 0 \Rightarrow \boxed{\nabla_{\mu} G^{\mu\nu} = 0}$ if $\Lambda = \text{constant}$.

The cosmological constant can be moved to the right side as the **vacuum stress-energy tensor**:

$$T^{\mu\nu}_{(vac)} = -\frac{\Lambda c^4}{8\pi G} g^{\mu\nu} = \rho_{vac} c^2 g^{\mu\nu} \text{ where } \rho_{vac} = \text{vacuum energy density} = \frac{\Lambda c^2}{8\pi G} \approx 0.7\times 10^{-26} kg/m^3 \ll 10^{-7}\text{kg/m}^3.$$

Then: $\boxed{G^{\mu\nu} = 8\pi G\left(T^{\mu\nu} + T^{\mu\nu}_{vac}\right)} \equiv 8\pi G T^{\mu\nu}_{all}$; Alternate: $\boxed{R^{\mu\nu} = 8\pi G\left(T^{\mu\nu}_{all} - g^{\mu\nu} T_{all}\right)}$ where $T \equiv g_{\mu\nu} T^{\mu\nu} = T^{\mu}_{\mu}$.

Note: $T^{\mu\nu}_{vac} - \frac{1}{2} g^{\mu\nu} T_{vac} = \left(-\Lambda g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} 4\Lambda\right) / (8\pi G) = \frac{\Lambda g^{\mu\nu}}{8\pi G}$.

Einstein equation is 10 nonlinear, usually coupled, 2nd-order differential equations to solve for the metric!

Symmetry with regard to coordinate changes $\to$ conservation of energy and momentum (Noether's Theorem).

Alternate Form of Einstein's equation: $\boxed{R^{\mu\nu} = \kappa\left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right) + \Lambda g^{\mu\nu}}$.

Spherical Surface of Radius R

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta, \text{ other } \Gamma^{\mu}_{\alpha\beta} = 0$$

$$R_{\theta\phi\theta\phi} = -R_{\theta\phi\phi\theta} = -R_{\phi\theta\theta\phi} = R_{\phi\theta\phi\theta} = r^2\sin^2\theta, \text{ other } R_{\alpha\beta\mu\nu} = 0, R_{\theta\theta} = 1, R_{\phi\phi} = \sin^2\theta, R_{\theta\phi} = R_{\phi\theta} = 0, \quad R = 2/r^2$$

Christoffel Symbol Properties

- Has 4x4x4 = 64 symbols
- Symmetric: $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$; therefore 40 independent symbols, but only 10 are unique
- Unique symbols are tt, tr, tθ, tϕ, rr, rθ, rϕ, $\theta\theta$, $\theta\phi$, $\phi\phi$.
- $\Gamma^{\lambda}_{\mu\nu} = 0$ for $\mu \neq \nu \neq \lambda$.

  $$\Gamma^{t}_{rt} = \Gamma^{t}_{tr} = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}.$$

Covariant/Absolute Derivative

Curvilinear coordinate system unit vectors change over space: $\boxed{\partial_\mu e_\alpha \equiv \Gamma^{\nu}_{\mu\alpha} e_\nu}$.

Vector A: $dA = d\left(A^\mu e_\mu\right) = \left(\partial_\sigma A^\mu dx^\sigma\right) e_\mu + A^\mu \partial e_\alpha dx^\alpha = \boxed{\left(\partial_\sigma A^\mu + \Gamma^{\mu}_{\alpha\nu} A^\nu\right) e_\mu dx^\alpha \equiv \left(\nabla_\alpha A^\mu\right) e_\mu dx^\alpha}$.

Covariant or absolute derivative: $\boxed{\nabla_\alpha A^\mu \equiv A^\mu_{,\alpha} + \Gamma^{\mu}_{\alpha\nu} A^\nu \equiv A^\mu_{;\alpha}}$. Covariant vector: $\boxed{\nabla_\alpha A_\mu \equiv A_{\mu,\alpha} - \Gamma^{\nu}_{\alpha\mu} A_\nu \equiv A_{\mu;\alpha}}$.

Tensor of rank 2:

$$\boxed{\begin{aligned} A_{\mu\nu;\lambda} &= A_{\mu\nu,\lambda} - \Gamma^{\alpha}_{\mu\lambda} A_{\alpha\nu} - \Gamma^{\alpha}_{\nu\lambda} A_{\mu\alpha} \\ A^{\mu}_{\nu;\kappa} &= A^{\mu}_{\nu,\kappa} + \Gamma^{\mu}_{\alpha\kappa} A^{\alpha}_{\nu} - \Gamma^{\alpha}_{\nu\kappa} A^{\mu}_{\alpha} \\ A^{\mu\nu}_{;\kappa} &= A^{\mu\nu}_{,\kappa} + \Gamma^{\mu}_{\alpha\kappa} A^{\alpha\nu} + \Gamma^{\nu}_{\alpha\kappa} A^{\mu\alpha} \end{aligned}}$$

Tensor of rank 3: $\boxed{\nabla_\alpha T^{\mu\nu}_{\sigma} = \partial_\alpha T^{\mu\nu}_{\sigma} + \Gamma^{\mu}_{\alpha\beta} T^{\beta\nu}_{\sigma} + \Gamma^{\nu}_{\alpha\delta} T^{\mu\delta}_{\sigma} - \Gamma^{\gamma}_{\alpha\sigma} T^{\mu\nu}_{\gamma}}$.

Solving for the Metric

1. Use symmetry to define a coordinate system as completely as possible.
2. Set up a trial metric with as few undetermined coefficients as possible.
3. Substitute the trial metric into the Einstein equation.
4. Solve the resulting differential equations for the unknown metric components.

Spherical Symmetry

Trial metric: $ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$.

The result is the Schwarzschild metric given below.

Empty-Space Metric

$$g_{\mu\nu} = \frac{dr^2 \delta_{\mu\nu}}{dx^\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ where } dr^2 = dr^\mu dr_\mu = -c^2dt^2 + dx^2 + dy^2 + dz^2 \; .$$

Equation of Motion/Geodesic

The equation of a "free" particle follows a "straight line" in curved space whose curvature is caused by the presence of energy/mass. Such a "straight line" is called a geodesic. A "free particle" is free of non-gravitational interactions.

The geodesic equation or equation of motion for a particle with rest mass in a local inertial frame is $\boxed{\frac{d^2x^\mu}{d\tau^2} = \frac{dU^\mu}{d\tau} = 0}$.

In another frame:

$$0 = \frac{dU^{\bar{\alpha}}}{d\tau} = \frac{d}{d\tau}\left(\frac{\partial x^{\bar{\alpha}}}{\partial x^\mu}U^\mu\right) = \frac{\partial x^{\bar{\alpha}}}{dx^\mu}\frac{dU^\mu}{d\tau} + U^\mu \frac{d}{d\tau}\left(\frac{\partial x^{\bar{\alpha}}}{\partial x^\mu}\right) = \frac{\partial x^{\bar{\alpha}}}{dx^\mu}\frac{dU^\mu}{d\tau} + U^\mu \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\bar{\alpha}}}{\partial x^\nu}\right)\frac{dx^\nu}{d\tau} = \frac{\partial x^{\alpha}}{dx^\mu}\frac{dU^\mu}{d\tau} + U^\mu \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\bar{\alpha}}}{\partial x^\nu}\right)U^\nu.$$

$$\text{Multiply by } \frac{\partial x^\beta}{\partial x^{\bar{\alpha}}} \text{ and change } \bar{\alpha} \text{ to } \alpha:$$

$$0 = \frac{\partial x^\beta}{\partial x^\alpha}\left[\frac{\partial x^\alpha}{dx^\mu}\frac{dU^\alpha}{d\tau} + U^\mu \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\alpha}{\partial x^\nu}\right)U^\nu\right] \equiv \delta^\beta{}_\mu \frac{dU^\mu}{d\tau} + U^\mu U^\nu \Gamma^\beta_{\mu\nu} = \boxed{\frac{dU^\beta}{d\tau} + U^\mu U^\nu \Gamma^\beta_{\mu\nu} = \frac{d^2x^\beta}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Gamma^\beta_{\mu\nu} = 0},$$

$$\text{where } \boxed{\frac{\partial x^\beta}{\partial x^\alpha}\frac{\partial x^\alpha}{dx^\mu} = \delta^\beta{}_\mu} \text{ and } \frac{\partial x^\beta}{\partial x^\alpha}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\alpha}{\partial x^\nu}\right) = \boxed{\frac{\partial x^\beta}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x^\mu \partial x^\nu} \equiv \Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\alpha\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)}.$$

$\boxed{\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}$ where $c^2d\tau^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}dx^\mu dx^\nu$. $\tau =$ proper time.

This can be rewritten in terms of observer's time coordinate, $t \equiv x^0$: $\boxed{\frac{d^2x^\mu}{dt^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}\frac{dx^\mu}{dt}}$.

If the particle's velocity is small the equation reduces to $\frac{d^2x^n}{dt^2} = -\Gamma^n_{00}$ where $n = (1,2,3)$.

Other forms of the geodesic:

$$\boxed{0 = \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}.$$

$$\boxed{\frac{d^2x^\gamma}{d\tau^2} = -g^{\gamma\alpha}\left(\partial_\nu g_{\alpha\mu} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\right)u^\mu u^\nu}.$$

For **free space**: $g_{\mu\nu} = \text{diagonal}(-1,1,1,1)$. So, $\partial_\mu g_{\alpha\beta} = 0$; then $0 = \frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) \to g_{\mu\nu}\frac{dx^\nu}{d\tau} = \text{constant}.$

So, the velocity is constant, as it should be for free space (except for the object being considered).

Newtonian (weak-field) Limit

Conditions

1. Particle velocity is small compared to c.
2. Gravitational field is weak; a small perturbation of flat space.
3. Gravitational field is static, constant in time.

Condition 1 requires that $\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$. This reduces the geodesic equation to $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0.$

Condition 2 requires that $\boxed{g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}}$ for $\left|h_{\mu\nu}\right| \ll 1$, $h_{\mu\nu} = \mathrm{h}_{\nu\mu}$; $\mathrm{g}^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$.

Condition 3 yields $\Gamma^\mu_{00} = -\frac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00}$.

Then $\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}$ and $\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{dt}{d\tau}\right)^2$.

Since $\partial_0 h_{00} = 0$ then $\frac{d^2t}{d\tau^2} = 0$ and $\frac{dt}{d\tau} =$ constant. Since $\eta_{ij} =$ 3x3 identity matrix: $\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\partial_i h_{00}$.

This corresponds to the Newton gravity equation if $h_{00} = -2\Phi$ and $g_{00} = -\left(c^2 + 2\Phi\right)$.

For a point particle or outside a spherical mass M: $\Phi(r) = -G\frac{M}{r}$.

Metric is $ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + \frac{2GM}{r}\right)d\vec{r}^{\,2}$. (Schwarzschild metric to first order.)

Correction to Force of Gravity

Newton: $F = ma = \frac{GmM}{r^2}$; General Relativity: $F = \frac{GmM}{r^2}\left(1 + \frac{v^2}{c^2}\right)$.

The correction is ~2 parts per billion!
The correction for the oblateness of the Earth is ~1 part in a thousand,l
a million times the GR correction.
However both corrections are important for GPS positioning.

Newtonian limit of cosmological constant

Work by Nowakowski shows that the cosmological constant in the Newtonian limit corresponds to

$$\Phi(r) = -G\frac{M}{r} - \frac{1}{6}\Lambda r^2 \text{ or a force on mass m is } F(r) = -m\frac{d\Phi(r)}{dr} = -G\frac{mM}{r^2} + \frac{m}{3}\Lambda r.$$

A cosmological constant adds in a repulsive gravitational force, which could be the cause of an expanding universe. Of course, the repulsive force has to be very small for r of solar-system size.

Spherical Body of Mass-energy M (Schwarzschild Metric)(no charge or spin)

$$g_{\mu\nu}=\begin{bmatrix} -c^2\left(1-\frac{r_s}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{r_s}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix},$$

where $r_s=\frac{2GM}{c^2}=$ **Schwarzschild radius**. ($r_s=0=$ flat spacetime.)

"The **Schwarzschild radius** (sometimes historically referred to as the **gravitational radius**) is the radius of a sphere such that, if all the mass of an object were to be compressed within that sphere, the escape velocity from the surface of the sphere would equal the speed of light. If a stellar remnant were to collapse to or below this radius, light could not escape and the object is no longer directly visible outside, thereby forming a black hole."

A more compact way to write it is:

$$ds^2=-c^2\left(1-\frac{r_s}{r}\right)dt^2+\left(1-\frac{r_s}{r}\right)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2.$$

"The **Schwarzschild radius of the Earth is roughly 8.9 mm, while the Sun, which is 3.3×10^5 times as massive has a Schwarzschild radius of approximately 3.0 km**. Even at the surface of the Earth, the corrections to Newtonian gravity are only one part in a billion. The ratio only becomes large close to black holes and other ultra-dense objects such as neutron stars. The Schwarzschild metric is a solution of Einstein's field equations in empty space, meaning that it is valid only *outside* the gravitating body. That is, for a spherical body of radius *R* the solution is valid for *r* > *R*. To describe the gravitational field both inside and outside the gravitating body the Schwarzschild solution must be matched with some suitable interior solution at *r* = *R*, such as the interior Schwarzschild metric."

Interior Schwarzschild Metric

$$g_{\mu\nu}=\begin{bmatrix} -\frac{c^2}{4}\left(3\sqrt{1-\frac{r_s}{r_g}}-\sqrt{1-\frac{r^2r_s}{r_g^3}}\right)^2 & 0 & 0 & 0 \\ 0 & \left(1-\frac{r^2r_s}{r_g^3}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix},$$

where $r_s=\frac{2GM}{c^2}=$ Schwarzschild radius and $r_g=r$ coordinate at the body's surface measured at ∞.

Assumed that it holds an incompressible fluid of constant density, $\rho=M/\left(\frac{4\pi}{3}r_g^{\ 3}\right)$, with zero pressure at the surface.

It matches the Schwarzschild Metric at r=r$_g$.

Schwarzschild Geodesic Equation (Equation of Motion)

The geodesic equation is $0=\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right)-\frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$. Note that $g_{\mu\nu}$ is time independent and diagonal, so

$$\boxed{0=g_{\mu\mu}\frac{d^2x^\mu}{d\tau^2}-\frac{1}{2}\partial_\mu g_{\alpha\alpha}\left(\frac{dx^\alpha}{d\tau}\right)^2}.$$

Time Component (μ= 1, x^1 = t)

$$g_{11}\frac{d^2t}{d\tau^2}=0\Rightarrow -g_{11}\frac{dt}{d\tau}=\boxed{c^2\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}=e}=\text{ constant.}$$

This e is the relativistic energy per unit mass as measured at ∞:

$$\text{At } r=\infty, \boxed{e=c^2\frac{dt}{d\tau}=\frac{c^2p^1}{m}}; \frac{dt}{d\tau}=\text{object's 4-velocity at }\infty.$$

Clock at rest ($dr=d\theta=d\phi=0$):

$$\text{Proper-time interval}=\Delta\tau=\int d\tau=\int\sqrt{-ds}=\int_{t_1}^{t_2}\sqrt{\left(1-\frac{2GM}{r}\right)dt^2}=\sqrt{\left(1-\frac{2GM}{r}\right)}\int_{t_1}^{t_2}dt=\boxed{\sqrt{\left(1-\frac{2GM}{r}\right)}\Delta t=\Delta\tau}.$$

Gravitational red shift:

$$\text{For } r_L>r_S: \frac{\lambda_{r_L}}{\lambda_{r_S}}=\frac{\sqrt{1-2GM/r_L}}{\sqrt{1-2GM/r_S}}\approx\left(1-\frac{2GM}{r_L}\right)\left(1-\frac{2GM}{r_S}\right)\underset{2GM\le r}{\Rightarrow}1+\frac{GM}{r_S}-\frac{GM}{r_L}\underset{\Delta r\le r_L}{\Rightarrow}\boxed{1+\frac{GM}{r_s^{\,2}}\left(r_L-r_s\right)\approx\frac{\lambda_{r_L}}{\lambda_{r_S}}}.$$

ϕ Component (μ= 4, x^4 =ϕ)

$$0=\frac{d}{d\tau}\left(g_{\varphi\nu}\frac{dx^\nu}{d\tau}\right)-\frac{1}{2}\partial_\phi g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}.$$

$g_{\mu\nu}$ is t and ϕ independent and diagonal, so $g_{\phi\phi}\frac{d^2\phi}{d\tau^2}=0\Rightarrow g_{\phi\phi}\frac{d\phi}{d\tau}=r^2\sin^2\theta\frac{d\phi}{d\tau}=\ell=\text{constant.}$

Choose spherical coordinates such that the orbit is on the equator; i.e., $\theta=\frac{\pi}{2}$ and $\sin\theta=1$.

Then $\boxed{\ell_z=r^2\frac{d\phi}{d\tau}=r^2\omega=\frac{L}{m}=}$ relativistic angular momentum/unit-mass.

θ Component (μ= 3, x^3 =θ)

$$0=\frac{d}{d\tau}\left(g_{\theta\nu}\frac{dx^\nu}{d\tau}\right)-\frac{1}{2}\partial_\theta g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=\frac{d}{d\tau}\left(g_{\theta\theta}\frac{d\theta}{d\tau}\right)-\frac{1}{2}\partial_\theta\left[g_{\theta\theta}\left(\frac{d\theta}{d\tau}\right)^2+g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2\right].$$

Or $0=\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right)-\frac{1}{2}\partial_\theta\left[r^2\left(\frac{d\theta}{d\tau}\right)^2+r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]=\frac{d}{d\tau}\left(r^2\frac{d\theta}{d\tau}\right)-\frac{1}{2}\partial_\theta\left[r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right].$

Or $\boxed{0=r^2\frac{d^2\theta}{d\tau^2}+2r\frac{dr}{d\tau}\frac{d\theta}{d\tau}-r^2\sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2}.$

Note that $\theta=\frac{\pi}{2}$ and $\cos\theta=0$ is a solution to this equation. So, $\frac{d\theta}{d\tau}=0$ means that the **orbit is planar**.

r Component (μ= 2, x^2 =r)

$$0=g_{rr}\frac{d^2r}{d\tau^2}-\frac{1}{2}\partial_r g_{\alpha\alpha}\left(\frac{dx^\alpha}{d\tau}\right)^2$$. All metric components depend on r.

Easy way: Use $-1=u\cdot u=g_{\mu\nu}u^\mu u^\nu$, $\boxed{e\equiv\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}}$, $\boxed{\ell\equiv r^2\sin^2\theta\frac{d\phi}{d\tau}}$ and $\theta=\frac{\pi}{2}\,(planar)$:

$$-1=g_{tt}\left(\frac{dt}{d\tau}\right)^2+g_{rr}\left(\frac{dr}{d\tau}\right)^2+g_{\theta\theta}\left(\frac{d\theta}{d\tau}\right)^2+g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2=$$

$$=-\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2+\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2+r^2\left(\frac{d\theta}{d\tau}\right)^2+r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\Rightarrow$$

$$\boxed{1=\left(1-\frac{2GM}{r}\right)^{-1}e^2-\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2-\frac{\ell^2}{r^2}}.$$

The solution is $\boxed{\bar{E}=\bar{K}+\bar{V}}$ or $\boxed{\frac{1}{2}\left(e^2-1\right)=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2-\frac{GM}{r}+\frac{\ell^2}{2r^2}-\frac{GM\,\ell^2}{r^3}}.$,

Where
$$\boxed{\begin{aligned}&\bar{E}\equiv\frac{1}{2}\left(e^2-1\right)=\text{effective conserved energy per unit mass,}\\&\bar{K}\equiv\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2=\text{effective radial kinetic energy per unit mass and}\\&\bar{V}\equiv-\frac{GM}{r}+\frac{\ell^2}{2r^2}-\frac{GM\,\ell^2}{r^3}=\text{effective potential energy per unit mass}\end{aligned}}.$$

The last term in $\bar{V}$ is not in Newtonian gravity.

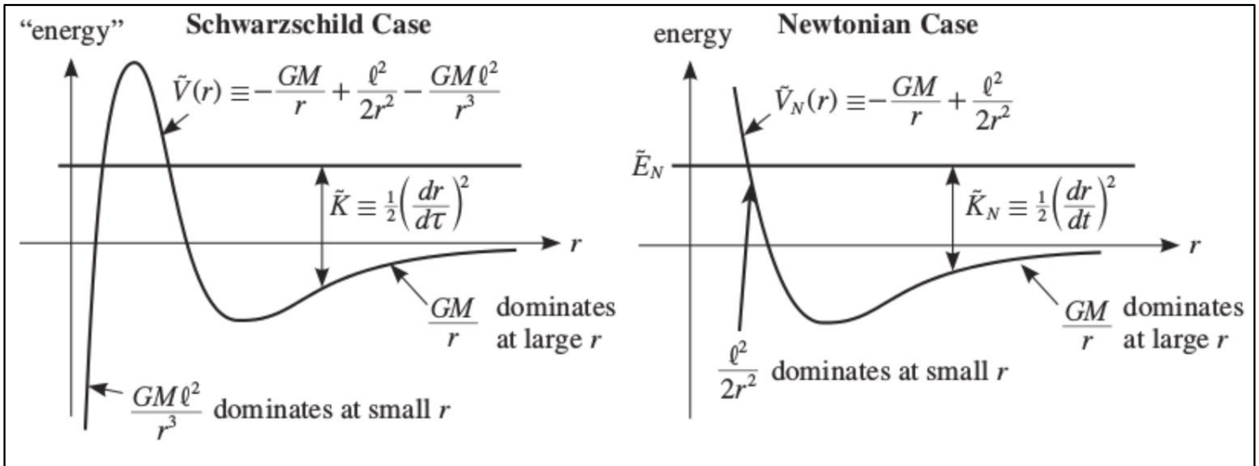


FIG. 10.1 These graphs show the effective potential-energy-per-unit-mass functions governing radial motion in the Schwarzschild case (left) and Newtonian case (right).

Circular Motion:

$$0 = \frac{d\bar{V}(r)}{dr} = \frac{d}{dr}\left[-\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}\right] = \frac{GM}{r_c^2} - \frac{\ell^2}{r_c^3} + \frac{3GM\ell^2}{r_c^4}.$$

Let $w_c \equiv \frac{1}{r_c} \Rightarrow 3GM\ell^2 w^2 - \ell^2 w + GM = 0$. Solve quadratic for w_c:

$\therefore \frac{1}{w_c} = \boxed{r_c = \frac{6GM}{1 \pm \sqrt{1 - 12(GM/\ell)^2}}}$, 2 circular orbits, one stable (+) and one unstable (-). (Newtonian: $r_c = \frac{\ell}{\sqrt{GM}}$).

Stable orbits: $r > 6GM$; Neutron stars: $r \approx 5GM$; Black holes: $r = 0$.

Radial Acceleration:

Do $\frac{d}{d\tau}$ on $\frac{1}{2}(e^2 - 1) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3} \Rightarrow$

$\boxed{a_r = \frac{d^2 r}{d\tau^2} = -\frac{GM}{r^2} + \frac{\ell^2}{r^3} - \frac{3GM\ell^2}{r^4}}$; Newtonian has only the first two terms. Radial motion $(\ell = 0)$: $a_r = -GM/r$.

Radial Distance $(dt = d\theta = d\phi = 0)$:

$$\Delta s = \int ds = \int_{r_S}^{r_L} \frac{dr}{\sqrt{1 - 2GM/r}} = \left[r\sqrt{1 - \frac{2GM}{r}} + 2GM \tanh^{-1}\sqrt{1 - \frac{2GM}{r}}\right]_{r_S}^{r_L} \underset{2GM \leq r}{\Rightarrow} \boxed{\Delta s \approx r_L - r_S + GM \ln\frac{r_L}{r_S}}.$$

4-velocity of Object at Rest

$$-1 = u \cdot u = g_{tt} u^t u^t = -\left(1 - \frac{2GM}{r}\right) du^2 \Rightarrow \boxed{u = \left(1 - \frac{2GM}{r}\right)^{-1/2}}; \text{ GE: } \frac{d^2 r}{d\tau^2} = -\frac{GM}{r}.$$

TABLE 14.1 This table summarizes the equations of motion for particles in the Schwarzschild equatorial plane.

	Particles with mass $m > 0$	Particles with mass $m = 0$ (e.g., photons)
Conserved quantities	$e = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau},\quad \ell = r^2\frac{d\phi}{d\tau}$	$b \equiv \frac{\ell}{e} = \frac{r^2}{(1-2GM/r)}\frac{d\phi}{dt}$
Equation for $d\tau^2$	$d\tau^2 = \left(1-\frac{2GM}{r}\right)dt^2 - \left(1-\frac{2GM}{r}\right)^{-1}dr^2 - r^2 d\phi^2$	$0 = \left(1-\frac{2GM}{r}\right)dt^2 - \left(1-\frac{2GM}{r}\right)^{-1}dr^2 - r^2 d\phi^2$
Equation for $dr/d\tau$	$\Rightarrow \frac{dr}{d\tau} = \pm\sqrt{e^2 - \left(1-\frac{2GM}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)}$	$\Downarrow$
Equation for dr/dt	$\Rightarrow \frac{dr}{dt} = \pm\left(1-\frac{2GM}{r}\right)\sqrt{1-\frac{1}{e^2}\left(1-\frac{2GM}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)}$	$\Rightarrow \frac{dr}{dt} = \pm\left(1-\frac{2GM}{r}\right)\sqrt{1-\left(1-\frac{2GM}{r}\right)\frac{b^2}{r^2}}$
Equation for $d\phi/dt$	$\Rightarrow \frac{d\phi}{dt} = \frac{\ell}{er^2}\left(1-\frac{2GM}{r}\right)$	$\Rightarrow \frac{d\phi}{dt} = \frac{b}{r^2}\left(1-\frac{2GM}{r}\right)$

Photon Motion in Schwarzschild Space

For a **photon: m_0=0 and ds^2=0**, so use $m\neq 0$ equations in combinations that are well defined for $m\to 0$:

Define impact parameter = $b \equiv \dfrac{\ell}{e} = \dfrac{r^2(d\phi/d\tau)}{(1-2GM/r)(dt/d\tau)} = \boxed{r^2\left(1-\dfrac{2GM}{r}\right)^{-1}\dfrac{d\phi}{dt} = b}$; flat spacetime: $r^2\dfrac{d\phi}{dt}$.

Photon Radial Motion (in $\theta=\pi/2$ plane): $0 = ds^2 = -\left(1-\dfrac{2GM}{r}\right)dt^2 + \left(1-\dfrac{2GM}{r}\right)^{-1}dr^2 + r^2 d\phi^2.$

Divide by $\left(1-\dfrac{2GM}{r}\right)dt^2$:

$$1 = \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 + r^2\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{d\phi}{dt}\right)^2 = \left(1-\frac{2GM}{r}\right)^{-2}\left(\frac{dr}{dt}\right)^2 + \frac{b^2}{r^2}\left(1-\frac{2GM}{r}\right),$$

using the $d\phi/dt$ equation above. Divide both sides by b^2:

$\boxed{\dfrac{1}{b^2} = \left[\dfrac{1}{b}\left(1-\dfrac{2GM}{r}\right)^{-1}\dfrac{dr}{dt}\right]^2 + \dfrac{1}{r^2}\left(1-\dfrac{2GM}{r}\right)}$. Flat spacetime: $\dfrac{1}{b^2} = \left[\dfrac{1}{b}\dfrac{dr}{dt}\right]^2 + \dfrac{1}{r^2}$.

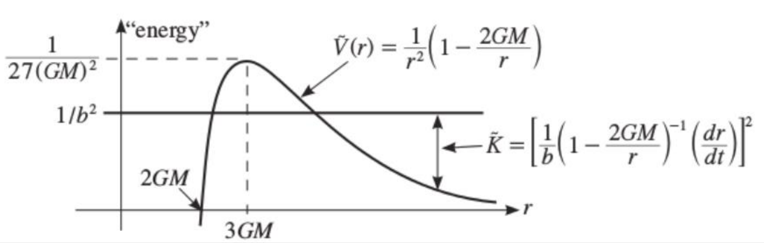


FIG. 12.2 This shows a graph of the effective potential energy function for the radial equation of motion for a photon. Larger values on the "energy" axis correspond to smaller impact parameters.

Schwarzschild Metric with Cosmological Constant Λ

$$g_{\mu\nu} = \begin{bmatrix} -c^2\left(1-\dfrac{2GM}{r}-\dfrac{r^2\Lambda}{3}\right) & 0 & 0 & 0 \\ 0 & \left(1-\dfrac{2GM}{r}-\dfrac{r^2\Lambda}{3}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix},$$

Then $\text{Relativistic Energy} = \boxed{e = \left(-\dfrac{2GM}{r}-\dfrac{r^2\Lambda}{3}\right)\dfrac{dt}{d\tau}}$.

$$\boxed{\bar{E}=\bar{K}+\bar{V}} \text{ or } \boxed{\frac{1}{2}\left(e^2-1\right)=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2-\frac{GM}{r}+\frac{\ell^2}{2r^2}-\frac{GM\ell^2}{r^3}-\frac{\Lambda}{6}\left(\ell^2+r^2\right)}.$$

Putting the cosmological constant, Λ, in Schwarzschild metric changes $\bar{V}$: $\boxed{\bar{V}=-\dfrac{GM}{r}+\dfrac{\ell^2}{2r^2}-\dfrac{GM\ell^2}{r^3}-\dfrac{\Lambda}{6}\left(\ell^2+r^2\right)}$.

Black Holes

A Black Hole is an object that does not have a surface outside of 2GM.

1. A clock at rest at $r = 2GM$ registers *no* time (i.e., when $r = 2GM$ and $dr = 0$, $d\theta = 0$, and $d\phi = 0$, then $d\tau^2 = (1 - 2GM/r)\,dt^2 = 0$).
2. (Related to this,) light emitted from rest at $r = 2GM$ is infinitely red-shifted when observed at any (larger) radius. Moreover, the redshift formula simply does not work for $r < 2GM$.
3. All particles (even photons) falling inward appear to a distant observer to "freeze" at $r = 2GM$ (see the expressions for dr/dt and $d\phi/dt$ in table 14.1).
4. Worst of all, g_{rr} goes to infinity at $r = 2GM$, implying that the derivative ds/dr of the radial distance s with respect to radial coordinate r diverges.

Event Horizon

A spherical surface with the Schwarzschild radius, **r_s=2GM**, is also called the Event Horizon. Any object inside cannot escape because the escape velocity would have to be greater than the speed of light.

Black Hole Density

Inside the Event Horizon the density is

$$\rho = \frac{mass}{volume} = \frac{M}{4\pi r_s^3/3} = \frac{3M}{4\pi r_s^3} = \frac{3M}{4\pi\left(2GM\right)^3} = \boxed{\frac{3}{32\pi G^3M^2} = \frac{3}{8\pi G r_s^2} = \rho}.$$

The larger or more massive the black hole the smaller the density.

Surface Gravity

$$\boxed{\kappa = \frac{1}{4M} = \frac{G}{2r_s}}.$$

Rotating-Black-Hole Surface Gravity

$$\kappa = \frac{1}{4M} - M\Omega_+^2 = 2\pi \mathrm{T}, \text{ where } \Omega_+ \equiv \text{ angular velocity of event horizon,}$$

$$\text{where } \Omega_+ = \frac{a}{r_+^2 + a^2}.$$

Black-Hole Thermodynamics

Units

$$\boxed{G = 7.426\times10^{-28}\, m/kg;\, M_\odot = 1.9891\times10^{30}\, kg;\, GM_\odot = 1477m;\, \hbar = 1.0546\times10^{-34}\, J\cdot s;\, k = 1.3807\times10^{-23}\, J/K.}.$$

Event-horizon temperature: $T = \dfrac{\hbar c^3}{8\pi k G M}$

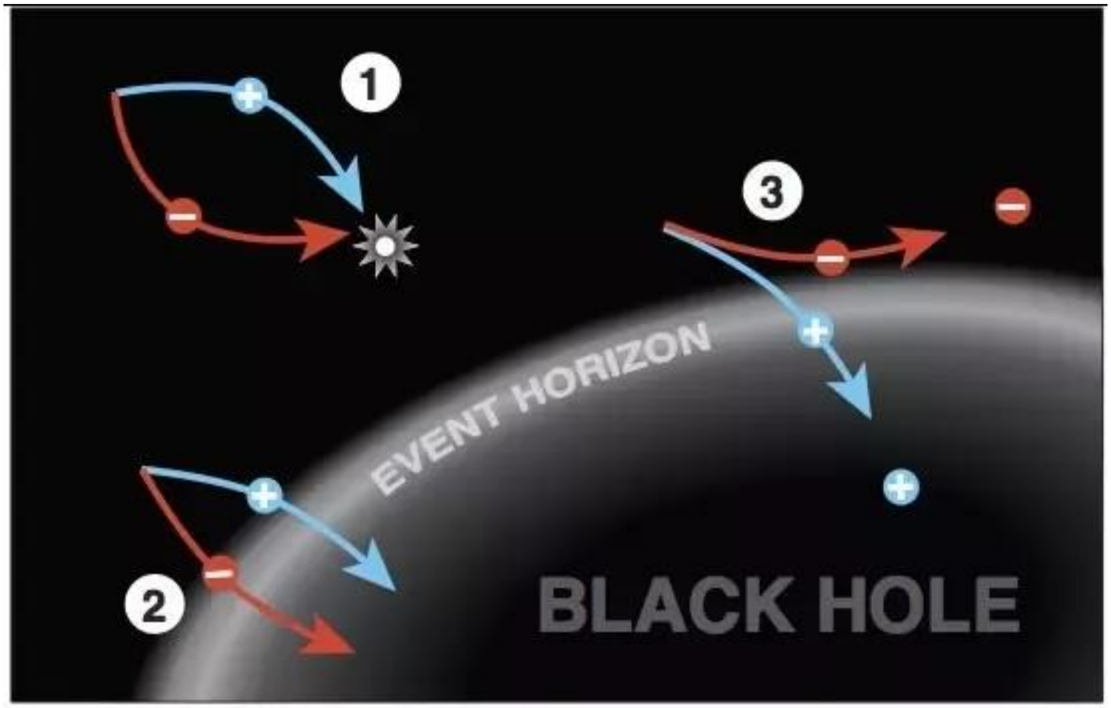
Entropy:

$$\boxed{S = \frac{kA}{4\ell_P^2}}, \text{ where } \boxed{A = 4\pi r_s^2 = 16\pi G^2 M^2}, k = \text{Boltzmann's constant \& } \boxed{\ell_P \equiv \sqrt{\frac{G\hbar}{c^3}}} = \text{Planck length.}$$

So, $\boxed{S = \dfrac{kc^3}{4G\hbar} A = \dfrac{\pi r_s^2 k c^3}{G\hbar} = \dfrac{4\pi G M^2 k c^3}{\hbar}}$; Kerr: $\boxed{S = \pi\left(r_+^2 + a^2\right)}$.

Hawking Radiation

Inside the event horizon a particle with mass can have negative energy/ mass: $\boxed{e = \left(1 - \dfrac{2GM}{r}\right)\dfrac{dt}{d\tau}}$.



Cosmic refugees. Virtual particles that escape destruction near a black hole (case 3) create detectable radiation but can't carry information.

Vacuum fluctuations create particle-antiparticle pairs. If one occurs near the event horizon, the negative-energy particle can pass through the event horizon and decrease the black-hole's energy, the decrease radiating away by means of the positive-energy particle. Since a photon is its own antiparticle and has zero mass, it is most likely that the particle-antiparticle pair is two photons. So the black-hole black-body radiation is photons.

Emitted Thermal Black-Body Radiation

$$\boxed{T = \frac{\hbar}{8\pi GMk} = \frac{\hbar}{4\pi r_s k}};\ \text{Kerr: } \boxed{T = \frac{1}{2\pi}\left(\frac{r_+}{r_+^2 + a^2} - \frac{1}{2r_+}\right)}.$$

$$T = \frac{\hbar}{8\pi kGM} = \frac{\hbar}{8\pi kGM_\odot}\frac{M_\odot}{M} = \boxed{\frac{61.7\times 10^{-9}K}{M/M_\odot} = T},\ M = M_\odot : T \approx 60\ nK.$$

Black-Hole Lifetime

Stefan-Boltzmann Law for energy radiation from a black-body:

$$\boxed{dE/dt = A\sigma T^4}, \text{ where } \sigma = \text{Stefan-Boltzmann constant.}$$

So, the mass-loss equation of a black hole is

$$\frac{dE}{dt} = -\frac{dM}{dt} = A\sigma T^4 = 4\pi r_s^2 \sigma \left(\frac{\hbar}{8\pi kGM}\right)^4 = 4\pi\left(2GM\right)^2 \sigma \left(\frac{\hbar}{8\pi kGM}\right)^4 =$$

$$= 4\pi\left(2G\right)^2 \sigma \left(\frac{\hbar}{8\pi kG}\right)^4 M^{-2} = \frac{\sigma\hbar^4}{256\pi^3 k^4 G^2} M^{-2}.$$

$$\text{So,} -\frac{256\pi^3 k^4 G^2}{\sigma\hbar^4}\int_M^0 M^2 dM = \frac{256\pi^3 k^4 G^2}{3\sigma\hbar^4} M^3 = \int_0^{\tau_{life}} dt = \tau_{life}.$$

$$\text{So, the lifetime of a black hole is } \boxed{\tau_{life} = \frac{256\pi^3 k^4}{3G\sigma\hbar^4}\left(GM\right)^3 = \frac{2048\pi^3 k^4}{3G\sigma\hbar^4} r_s^3}.$$

$$\tau_{life} = \left(1.095\times 10^{67}\, yr\right)\left(\frac{M}{M_\odot}\right)^3$$

Universe age = 13.82 x 10^9 years: $\begin{cases} M_{\min} = M_\odot \sqrt[3]{\left(13.82\times 10^9\right)/\left(\left(1.095\times 10^{67}\right)\right)} = 1.0807\times 10^{-19} M_\odot \\ = \left(1.0807\times 10^{-19} M_\odot\right)\left(1.98855\times 10^{30}\,\text{kg}/M_\odot\right) = \boxed{2.149\times 10^{11}\,\text{kg} = M_{\min}} \end{cases}$

Minimum black-hole radius: $r_{SunBlackHole} = 3.0\text{ km}, r_{\min} = \left(3.0\text{ km}\right)\left(1.0807\times 10^{-19}\right) = \boxed{3.2421\times 10^{-19}\,\text{km} = r_{\min}}$

Reissner-Nordstrom Metric for a Charged (Q), Non-Spinning Spherical Body of Mass M

$$g_{\mu\nu} = \begin{bmatrix} -c^2\left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \text{ where } \boxed{r_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4}}.$$

This reduces to the Schwarzschild metric when $Q = 0$.

Kerr Metric for a Spinning Black Hole (J) with a Spherical Event Horizon

$$g_{\mu\nu} = \begin{bmatrix} -c^2\left(1-\dfrac{r_s r}{\rho^2}\right) & 0 & 0 & -\dfrac{2r_s rac\sin^2\theta}{\rho^2} \\ 0 & \dfrac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\dfrac{2r_s rac\sin^2\theta}{\rho^2} & 0 & 0 & \left(r^2+a^2+\dfrac{r_s ra^2}{\rho^2}\sin^2\theta\right)\sin^2\theta \end{bmatrix},$$

where $\boxed{\rho^2 \equiv r^2 + a^2\cos^2\theta}$, $\boxed{\Delta \equiv r^2 - r_s r + a^2}$, $\boxed{r_s = \dfrac{2GM}{c^2}}$, $\boxed{a = \dfrac{J}{Mc}}$ and $J =$ spin.

This reduces to the Schwarzschild metric when $J = 0$.

In fact, astrophysical objects that might collapse to black holes will almost inevitably have nonzero angular momenta, indeed enough so that black holes formed by almost any conceivable process will have $a \approx GM$ (Bardeen, *Nature*, **226**, 1970, 64–65).

Physicists also strongly believed that a black hole will not form with $a \geq GM$: the high angular momentum of the collapsing object will lead it to spin off some material. This means that astrophysical black holes formed by collapse will almost certainly be Kerr black holes with $a \approx GM$.

Weak-Field Limit of the Kerr Metric

In Schwarzschild metric throw away terms in a^2/r^2 and use first term in binomial expansion of g_{rr}:

$$g_{\mu\nu} = \begin{bmatrix} -c^2\left(1-\dfrac{2GM}{r}\right) & 0 & 0 & -\dfrac{4GMac\sin^2\theta}{r} \\ 0 & \left(1+\dfrac{2GM}{r}\right) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -\dfrac{4GMac\sin^2\theta}{r} & 0 & 0 & r^2\sin^2\theta \end{bmatrix}.$$

Particle Orbits in Kerr Spacetime

Geodesic equations of motion: $\boxed{0 = \dfrac{d}{d\tau}\left(g_{\mu\nu}\dfrac{dx^\nu}{d\tau}\right) - \dfrac{1}{2}\partial_\mu g_{\alpha\beta}\dfrac{dx^\alpha}{d\tau}\dfrac{dx^\beta}{d\tau}}$.

$$\text{If } \mu = t: 0 = \frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau} + g_{t\phi}\frac{d\phi}{d\tau} \right) + 0 \Rightarrow \boxed{e \equiv -\left(g_{tt}\frac{dt}{d\tau} + g_{t\phi}\frac{d\phi}{d\tau} \right)},$$

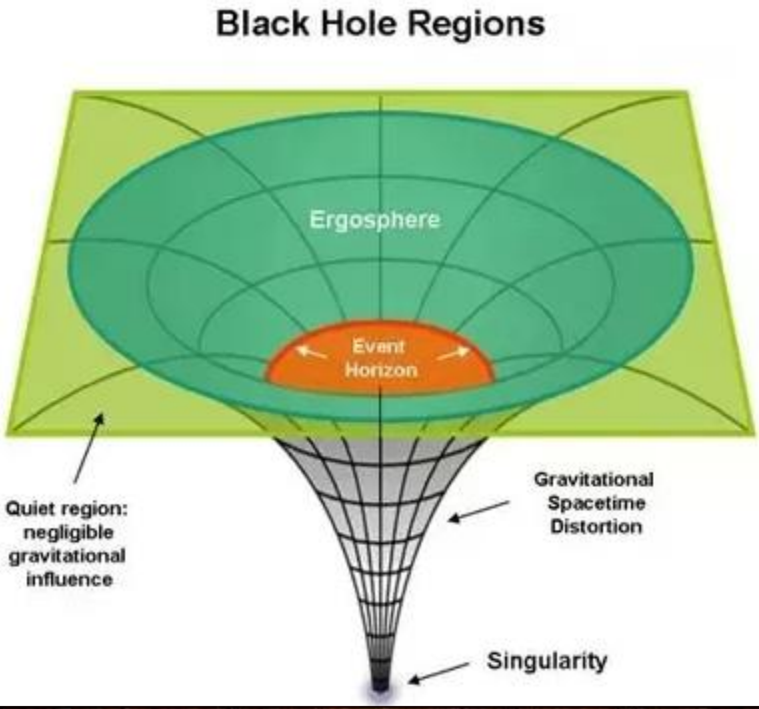
$$\text{If } \mu = \phi: 0 = \frac{d}{d\tau}\left(g_{\phi t}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau} \right) + 0 \Rightarrow \boxed{\ell \equiv g_{\phi t}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau}}.$$

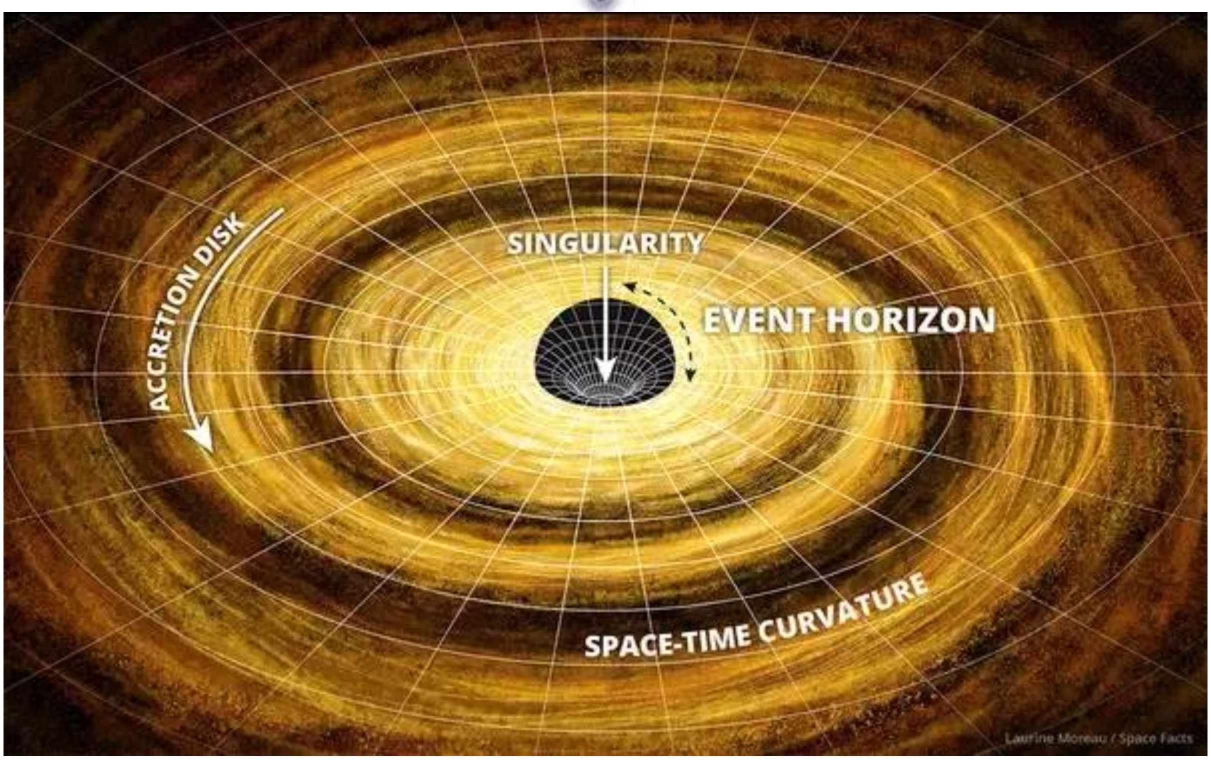
As $r \to \infty: g_{tt} \to -1,\ g_{t\phi} \ll g_{tt}$ and $g_{\phi\phi} \to r^2 \sin^2\theta$:

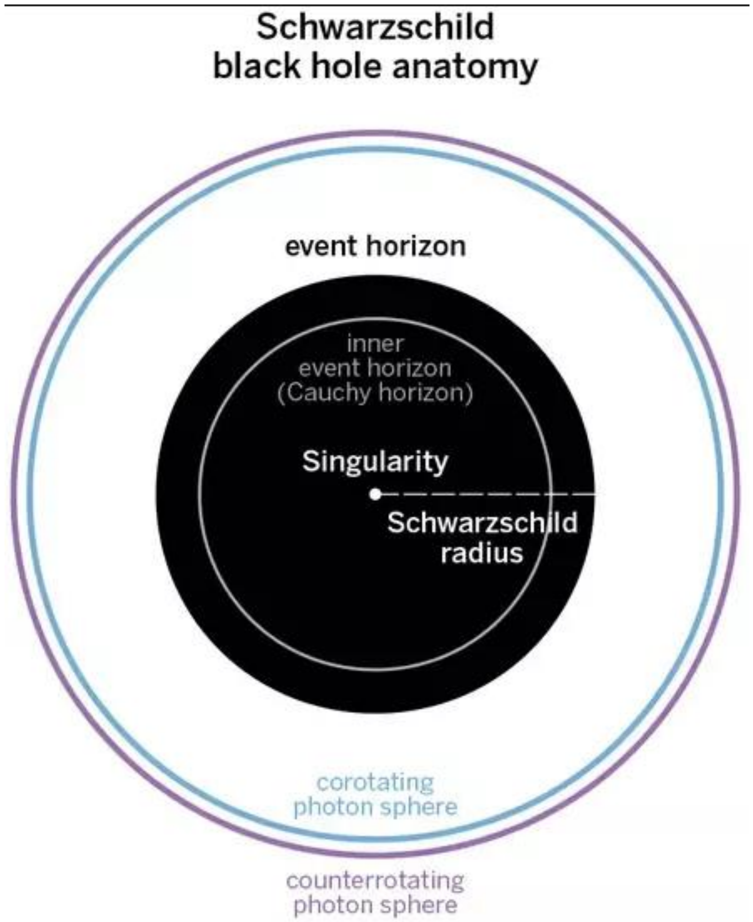
$e = \left(\frac{dt}{d\tau} \right)_\infty$ = relativistic energy per unit mass and

$\ell = \left(r^2 \sin^2\theta \frac{d\phi}{d\tau} \right)_\infty$ = angular-momentum z-component per unit mass.

Solving: $\boxed{\dfrac{dt}{d\tau}=\dfrac{g_{\phi\phi}e+g_{t\phi}\ell}{\left(g_{t\phi}\right)^2-g_{\phi\phi}g_{tt}}=\dfrac{g_{\phi\phi}e+g_{t\phi}\ell}{R^2\sin^2\theta}}$, $\boxed{\dfrac{d\phi}{d\tau}=-\dfrac{g_{\phi t}e+g_{tt}\ell}{\left(g_{t\phi}\right)^2-g_{\phi\phi}g_{tt}}=-\dfrac{g_{\phi t}e+g_{tt}\ell}{R^2\sin^2\theta}}$, where $\boxed{R^2\equiv r^2+a^2-2GMr}$.







Lastly, this is more or less what an actual black hole with an accretion disc would look like[1]:

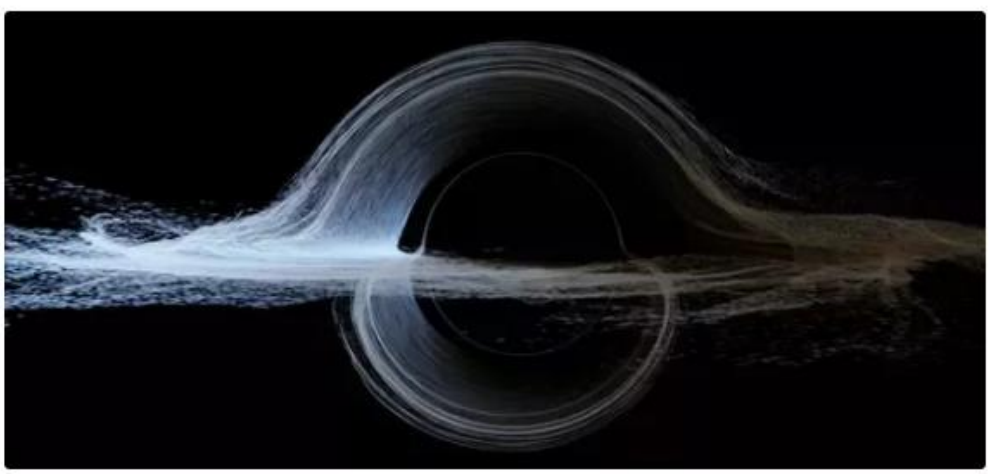


Image credit: James et al., licensed under CC BY-NC-ND 3.0

What you see here is the Doppler effect in action, where the light of the infalling material is being blueshifted where it spirals towards the observer, and redshifted where it spirals away from the observer. Furthermore, the part of the accretion disc behind the black hole is being warped above and below the black hole due to gravitational lensing, so we can effectively see material from behind the black hole.

Consider only orbits in the equatorial plane:

$$g_{\mu\nu} = \begin{bmatrix} -c^2\left(1-\frac{2GM}{r}\right) & 0 & 0 & -\frac{2GMac}{r} \\ 0 & \frac{r^2}{R^2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ -\frac{2GMac}{r} & 0 & 0 & r^2+a^2+\frac{2GMa^2}{r} \end{bmatrix},$$

For a particle of non-zero rest mass: $u \cdot u = -1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 + 2g_{t\varphi}\frac{dt}{d\tau}\frac{d\phi}{d\tau}$.

Then $\boxed{\tilde{E} \equiv \frac{1}{2}\left(e^2-1\right) = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{\ell^2 + a^2\left(1-e^2\right)}{2r^2} - \frac{GM\left(\ell - ea\right)^2}{r^3} \equiv K + V_{eff}\left(r\right)}$,

a "conservation-of-energy-like" equation.

For $a = 0: \tilde{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}$, the Schwarzschild value.

"Force-likes" equation: $\boxed{\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} + \frac{\ell^2 + a^2\left(1-e^2\right)}{2r^3} - \frac{GM\left(\ell - ea\right)^2}{r^4}}$.

Also, $\boxed{\frac{d\phi}{d\tau} = -\frac{g_{t\phi}e + g_{tt}\ell}{R^2} = \frac{2GMae + \left(1-2GM\right)\ell}{r\left(r^2 - 2GMr + a^2\right)}}$.

Solve the last two equations numerically, given $\boxed{M, a, \ell \text{ and } e}$.

For $\ell = 0: \frac{d\phi}{dt} = \frac{2GMae}{rR^2} = \frac{2GMae}{r\left(r^2 - 2GMr + a^2\right)^2} > 0 \text{ for } r > 2GM$, a gravitomagnetic deflection.

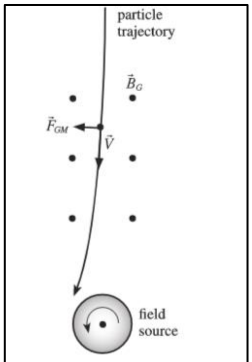


FIG. 37.1 Deflection of a particle falling inward with $\ell = 0$. Note that the particle is swept in the direction of the source's rotation.

"Dragging of inertial frames." Outward particle dragged opposite rotation.

Kepler's Third Law for Circular Orbits

Geodesic equation: $0 = \left(\frac{dt}{d\tau}\right)^2 \left(\partial_r g_{tt} + 2\partial_r g_{t\phi}\Omega + \partial_r g_{\phi\phi}\Omega^2\right)$ where $\Omega \equiv \frac{d\phi}{d\tau}$; solution: $\Omega = \frac{\sqrt{GM}}{\sqrt{GM}\,a \pm r^{3/2}}$.

$$T = \left|\frac{2\pi}{\Omega}\right| = \sqrt{\frac{4\pi^2}{GM}} r^{3/2} \pm 2\pi a \underset{a\to 0}{\to} \text{ Kepler's Third Law. } (\pm \text{ in orbit with/against source rotation}).$$

Innermost stable orbit: $\boxed{r^2 - 6GMr - 3a^2 \pm 8a\sqrt{GMr} = 0} \Rightarrow \boxed{r = 6GM}$ for $a = 0$ the Schwarzschild value.

For extreme value $a = GM : r = GM$ for co-rotating and $r = 9GM$ for counter-rotating orbit.

Kerr-Newman Metric for a Charged (Q), Spinning (J) Spherical Body of Mass M

$$g_{\mu\nu} = \begin{bmatrix} \frac{c^2}{\rho^2}\left(a^2\sin^2\theta - \Delta\right) & 0 & 0 & \frac{2ac}{\rho^2}\sin^2\theta\left(\Delta - r^2 - a^2\right) \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ \frac{2ac}{\rho^2}\sin^2\theta\left(\Delta - r^2 - a^2\right) & 0 & 0 & \left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]\frac{\sin^2\theta}{\rho^2} \end{bmatrix},$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\boxed{\Delta \equiv r^2 - r_s r + a^2 + r_Q^2}$, $r_s = \frac{2GM}{c^2}$, $a = \frac{J}{Mc}$ and $J =$ spin.

This reduces to the Schwarzschild metric when
$J = 0$ and $Q = 0$ and the Kerr metric when $Q = 0$ and the Reissner-Nordstrom metric when $J = 0$.

Infinite-Redshift Surface/Event Horizon

Schwarzschild: At $r = 2GM$: $g_{tt} = 0$: clocks at rest there measure zero proper time relative to clocks at infinity. There is **infinite redshift**. **Event Horizon**: An ingoing particle cannot escape.

Kerr metric for spinning source: Infinite-redshift surface encloses the event horizon, the **ergoregion**.

Proper time: $d\tau = \sqrt{-g_{tt}}\,dt$ between events separated by coordinate time dt.

Kerr black hole has two **infinite-redshift surfaces**: $\boxed{r = GM \pm \sqrt{(GM)^2 - a^2\cos^2\theta}}$ for $\boxed{a = \dfrac{J}{M} < GM}$.

At poles: $r = GM + \sqrt{(GM)^2 - a^2}$ and **at equator**: $r = 2GM$.

The event horizon is between those two surfaces, so only the outer one has meaning.

Consider a circular equatorial orbit: angular velocities: $\boxed{\Omega = \dfrac{2GMa}{r^3 + a^2 r + 2GMa^2} \pm \sqrt{\dfrac{r^2 + a^2 - 2GMr}{\left(r^2 + a^2 + 2GMa^2/r\right)^2}}}$.

Event Horizon: $0 = \left(g_{t\phi}\right)^2 - g_{tt}g_{\phi\phi} = \left(r^2 + a^2 - 2GMr\right)\sin^2\theta \equiv R^2\sin^2\theta.$

On equator: $0 = r^2 + a^2 - 2GMr \Rightarrow \boxed{r = GM \pm \sqrt{(GM)^2 - a^2}}$, Only the + one has meaning.

The metric on the event-horizon surface: $\boxed{ds^2 = \rho^2 d\theta^2 + \left(\dfrac{2GM}{\rho}\right)^2 \sin^2\theta\, d\phi^2}$ where $\rho^2 \equiv r^2 + a^2\cos^2\theta.$

Its surface area is $\boxed{A = 8\pi GMr}$.

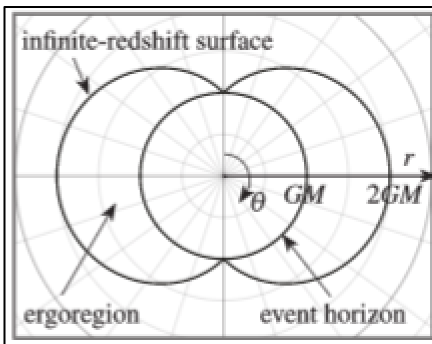


FIG.38.1 A polar $r(\theta)$ plot of the infinite-redshift and event horizon surfaces for an extreme Kerr black hole ($a = GM$). The r-coordinate of the event horizon is $r_+ = GM$: the equatorial r-coordinate of the infinite-redshift surface is $r_e = 2GM$.

The event horizon is a sphere only if a=0.

$$ds^2 = \rho^2 d\theta^2 + \left(\frac{2GM}{\rho}\right)^2 \sin^2\theta\, d\phi^2 \underset{a=0}{\rightarrow} r^2 d\theta^2 + \left(2GM\right)^2 \sin^2\theta\, d\phi^2, \text{ a sphere of radius } 2GM.$$

Photon sphere: A **photon sphere** is a spherical region of space where gravity is strong enough that photons are forced to travel in orbits. The radius of the photon sphere, which is also the lower bound for any stable orbit, is for a Schwarzschild black hole $dr = 0, ds = 0 \;\&\; d\theta = 0:$ Schwarzschild metric is $-(1 - r_s/r)dt^2 + r^2 \sin^2\theta d\phi = 0.$

$\therefore \boxed{\frac{d\phi}{dt} = \frac{\sqrt{1 - r_s/r}}{r\sin\theta}}$. The radial geodesic equation is

$\frac{d^2 r}{d\tau^2} + \Gamma^r_{\mu\nu}u^\mu u^\nu = 0$ or $\Gamma^r_{\mu\nu}u^\mu u^\nu = 0$. Evaluating $\Gamma^r_{\mu\nu}$ yields $\boxed{\frac{d\phi}{dt} = \sqrt{\frac{r_s}{2r^3\sin^2\theta}}}$.

For $\theta = \pi/2$: $\boxed{r_p = 3r_s/2 = 3GM}$ for $J = 0$.

For $J > 0$: A Kerr (spinning) black hole does not have spherical symmetry, but only an axis of symmetry, which has profound consequences for the photon orbits. A circular orbit can only exist in the equatorial plane, and there are two of them (prograde and retrograde), with different radii,

$$\boxed{r_\pm = r_k\left[1 + \cos\left(\frac{2}{3}\cos^{-1}\left(\frac{\pm J}{M}\right)\right)\right]}; 1 + \cos\left(\frac{2}{3}\cos^{-1}\left(\frac{0}{M}\right)\right) = \frac{3}{2},$$

$$\text{where } r_k = GM + \sqrt{(GM)^2 - \left(\frac{J}{M}\right)^2} \xrightarrow[J=0]{} 2GM = r_s.$$

J is the angular momentum. There exist other constant-radius orbits, but they have more complicated paths which oscillate in latitude about the equator.

Cosmic Censorship. Note that solutions to equation 38.7 for the event horizon radii fail to exist if $a > GM$, meaning that there are no event horizons in such a case. Now Kerr spacetime has a true geometric singularity (a place where the curvature of spacetime becomes infinite, analogous to $r = 0$ in Schwarzschild spacetime) where $r = 0$. If $a > GM$, there would be no event horizons surrounding this singularity, meaning that observers could visit this physical absurdity and send information back. This would raise a host of deep theoretical problems having to do with such issues as causality and self-consistency of the theory.

The **Cosmic Censorship Hypothesis** (first proposed by Roger Penrose in 1969) asserts that the gravitational collapse of a physically reasonable mass distribution can never produce such a "naked" singularity "unclothed" by an event horizon. As of this writing, this hypothesis remains unproven, but there is evidence to suggest that a Kerr spacetime with $a > GM$ would be unstable, meaning that it would spontaneously radiate gravitational waves until $a < GM$. Moreover, there are published arguments that suggest that any collapsing physical object with $a > GM$ would fragment into pieces before forming a black hole. We will therefore assume that the Cosmic Censorship Hypothesis is true and that all astrophysical black holes have $a < GM$ and thus have their singularities discreetly clothed by event horizons.

Cosmology

Observable Universe

Size: 12 Gly, ~10^{11} galaxies; Galaxies: Average size: ~50,000 ly, ~10^{11} stars; Milky Way: Size: 100,000 ly, ~3 x 10^{11} stars.

Solar System: Size: ~28,000 ly from Milky-Way center and 20 ly above MW central plane.

Distance to stars is determined by parallax, Cepheid variables and Type 1a supernovae.

Supernovae occur in a galaxy about every 300 years.

Galaxies are receding by Hubble's Law: $v \cong H_0 d$, where d = distance. H_0 =70.4±1.5 (km/s)/Mpc = $[13.9 \pm 0.3 \text{ Gy}]^{-1}$ in GRU.

Big Bang = $t_0 = H_0^{-1}$ = 13.9 Gy ago.

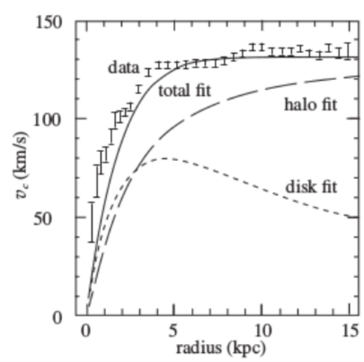
Sky black-body temperature = 2.725±0.001 K.

Opaque universe to ~380,000 years after Big Bang.

Universe is isotropic > 1 part in 100,000 and homogenous.

Composition of Universe

- Ordinary matter (protons, neutrons, electrons, etc.): 4.56±0.16%; density ≅ 4 x 10^{-28} kg/m3
- Radiation (photons and neutrinos) ≅0.0084%
- Non-baryonic dark matter 22.7±1.4% (cold WIMPs?)
- Dark Energy 72.8±1.6% (cosmological constant vacuum energy)

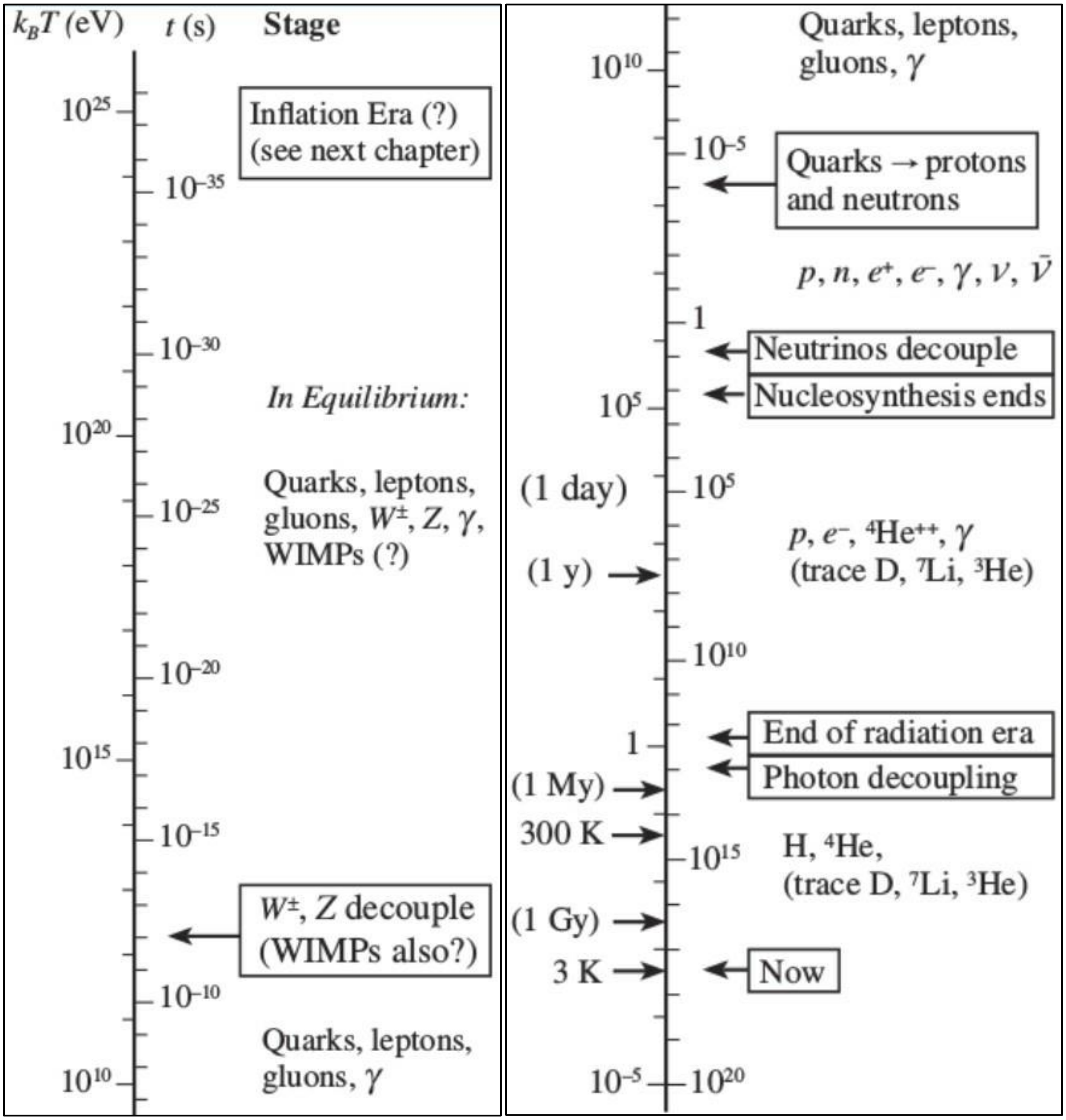


Metric for the Universe

Three metrics:

$$\boxed{\begin{gathered} ds^2 = -dt^2 + a^2(t)\left[d\bar{r}^2 + q^2(\bar{r})\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], a(t) \text{ specifies universe scale.} \\ \text{where } q(\bar{r}) = \begin{bmatrix} R\sin\left(\dfrac{\bar{r}}{R}\right) \\ \bar{r} \\ R\sinh\left(\dfrac{\bar{r}}{R}\right) \end{bmatrix} \text{ and } \bar{r} \text{ is comoving with the expanding universe.} \end{gathered}}.$$

Universe History



$$[a(t)]^2 - [a(t_s)]^2 = 2\sqrt{\Omega_r}\, H_0(t - t_s) \quad \text{(when radiation dominates)}$$

$$[a(t)]^{3/2} - [a(t_s)]^{3/2} = \tfrac{3}{2}\sqrt{\Omega_m}\, H_0(t - t_s) \quad \text{(when matter dominates)}$$

$$a(t) = a(t_s)e^{(t-t_s)H_0} \quad \text{(when vacuum energy dominates and } \Omega_k = 0\text{)}$$

where t_s is the time when the component in question starts to dominate.

$$\rho_c \equiv \frac{3H_0^2}{8\pi G} \tag{26.17}$$

and compare the present energy densities of matter, radiation, and the vacuum to this critical density by defining the unitless ratios

$$\Omega_m \equiv \frac{\rho_{m0}}{\rho_c}, \quad \Omega_r \equiv \frac{\rho_{r0}}{\rho_c}, \quad \Omega_v \equiv \frac{\rho_v}{\rho_c} \tag{26.18}$$

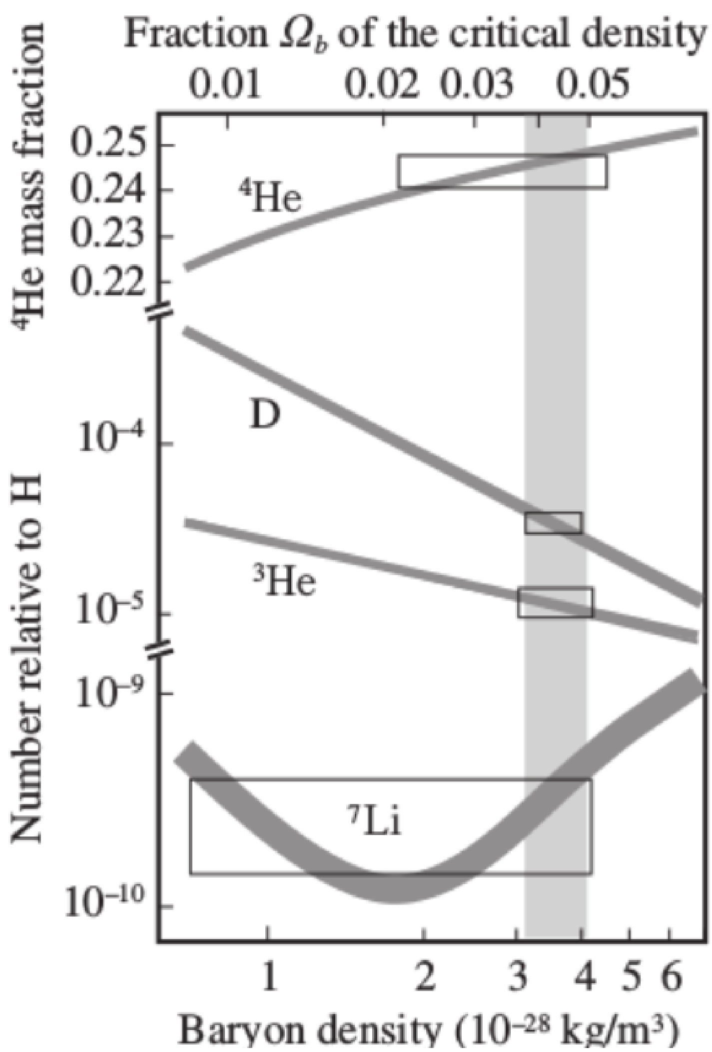


FIG. 28.2 This diagram shows the theoretical abundances of various nuclei as a function of the current density of baryonic matter. The vertical side of the box associated with each curve corresponds to the uncertainty in the measured value of that nucleus's abundance: the horizontal side thus corresponds to the range of baryon densities consistent with that abundance. The vertical gray bar specifies the range of baryon densities consistent with all measured abundances. This figure is adapted from Charbonnel, *Nature*, **415**, 2002, 27–29.

Cosmic Microwave Background (CMB) Fluctuations and Inflation

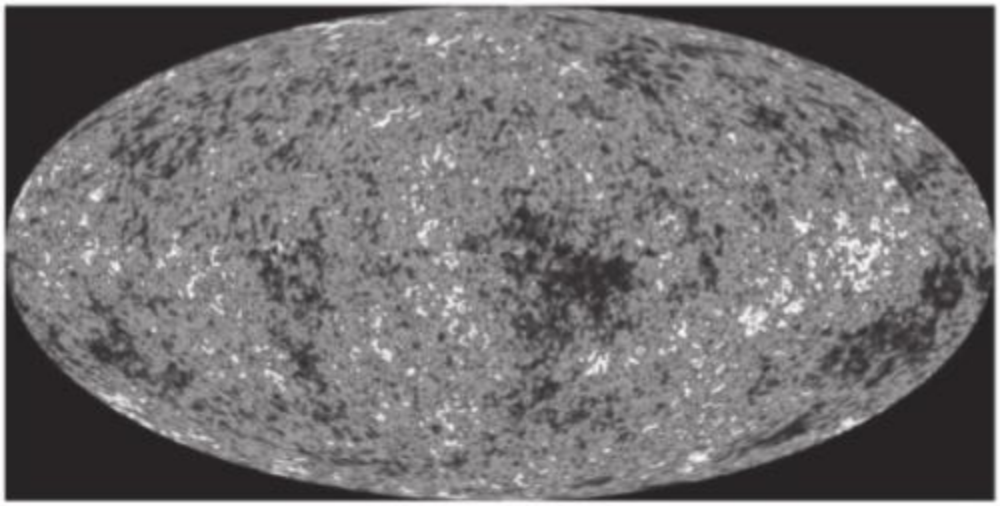


FIG. 29.1 This is a map of the CMB (corrected for the earth's motion and contributions from the galaxy). The fluctuations represent temperature changes on the order of tens of μK. **Credit:** NASA and the Wilkinson Microwave Anisotropy Probe (WMAP) team.

Isotropic to a few parts in 10^5 and universe is very nearly flat.

Inflation

The isotropy and flatness are explained by early universe rapid exponential expansion (inflation):

Vacuum dominated: $a(t) = a(t_s)\exp\left(\sqrt{\frac{8}{3}\pi G\rho}\,(t - t_s)\right)$ where t_s is the time when inflation started.

Grand Unified Theories (GUTs):

- Thermal energy = kT > $\approx 10^{15}$ GEV ~ 10^5 J: strong, weak & EM interactions are one.
- $\approx 10^{15}$ GEV > kT >$\approx$100 GeV: strong separate from electroweak interaction, a phase change.
- $\approx$100 GeV > kT: weak and EM interactions separate, a phase change.

Friedmann--Lemaître-Robertson-Walker Metric

"Consider a homogeneous, isotropic expanding or contracting universe that is path connected, but not necessarily simply connected."

$$g_{\mu\nu}=\begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & a(t)^2\dfrac{1}{1-kr^2} & 0 & 0 \\ 0 & 0 & a(t)^2 r^2 & 0 \\ 0 & 0 & 0 & a(t)^2 r^2\sin^2\theta \end{bmatrix}.$$

K = constant representing the curvature of space.

"This model is sometimes called the *Standard Model* of modern cosmology."

However, see Lambda-CDM model.

De Sitter Metric

"de Sitter space is the maximally symmetric vacuum solution of Einstein's field equations with a positive cosmological constant (corresponding to a positive vacuum energy density and negative pressure). When $n = 4$ (3 space dimensions plus time), it is a cosmological model for the physical universe; see de Sitter universe."

$$g_{\mu\nu}=\begin{bmatrix} -c^2\left(1-\dfrac{r^2}{\alpha^2}\right) & 0 & 0 & 0 \\ 0 & \left(1-\dfrac{r^2}{\alpha^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}.$$

r=α is a cosmological horizon.

Electromagnetism

In cgs units:

Four-Potential

$$A^\alpha=\left(\phi,\vec{A}\right);\boxed{\vec{E}=-\nabla\phi-\frac{1}{c}\partial_t\vec{A}},\boxed{\vec{B}=\nabla\times\vec{A}}.$$

Vector notation

$$\boxed{\nabla\cdot\vec{E}=4\pi\rho},\ \boxed{\nabla\times\vec{B}-\frac{1}{c}\partial_t\vec{E}=\frac{4\pi}{c}\vec{j}},\boxed{\nabla\cdot\vec{B}=0},\ \boxed{\nabla\times\vec{E}+\frac{1}{c}\partial_t\vec{B}=0}.$$

Lorentz Gauge

$$\partial_\alpha A^\alpha=0,\boxed{j^\alpha=\left(c\rho,j_x,j_y,j_z\right)}.\text{Define:}\boxed{\Box\equiv\frac{1}{c^2}\partial_t^2-\nabla^2}:\boxed{\Box A^\alpha=\frac{4\pi}{c}j^\alpha}.$$

Tensor notation

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}:$$

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu} = \frac{4\pi}{c}\left(c\rho, j_x, j_y, j_z\right), \boxed{\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0}.$$

Stress-Energy Tensor

Newtonian Gravity

$$\vec{g} = -\vec{\nabla}\Phi, \quad -\nabla \cdot g = -\nabla \cdot \left(-\vec{\nabla}\Phi\right) = \nabla^2\Phi = 4\pi G\rho, \text{ spherical mass } M : \vec{g} = \left(GM/r^2\right)\hat{r}.$$

The Riemann tensor in GR plays a similar role as $\partial_j \partial_k \Phi$ does in NG.

$$\text{Fluid: } \boxed{\rho = \rho_0 u^\mu u^\nu = T^{\mu\nu} = T^{\nu\mu} = \text{stress-energy tensor; } T^{ij} = \text{i-flux of j-momentum}}.$$

Conservation of fluid's energy $\Rightarrow \boxed{\partial_\mu T^{t\nu} = 0}$.

Perfect fluid (ideal gas) at rest in a LIF: $T^{\mu\nu} = \begin{bmatrix} \rho_0 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{bmatrix}$.

Arbitrary coordinates: $T^{\mu\nu} = \left(\rho_0 + p_0\right) u^\mu u^\nu + p_0 g^{\mu\nu}$.

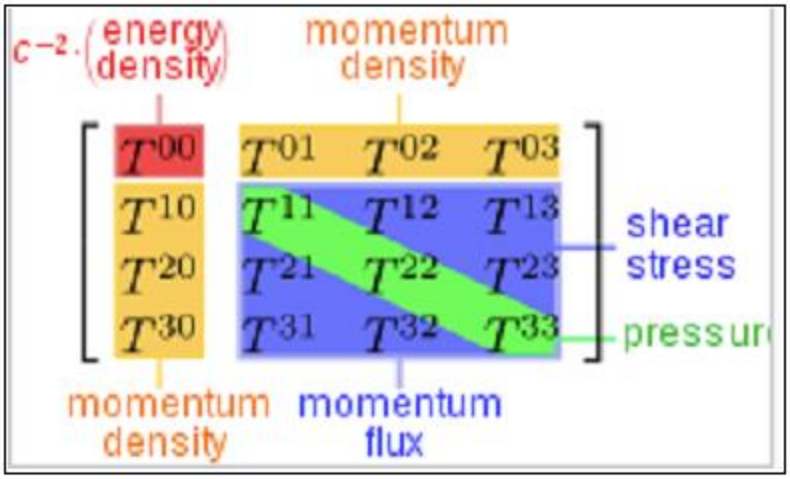
Conservation Law $\left(\nabla_\nu T^{\mu\nu} = 0\right)$:

$$\partial_\mu\left(\rho_0 u^\mu\right) + p_0 \partial_\mu u^\mu \text{ (eq. continuity); } \left(\rho_0 + p_0\right)\left(\partial_\mu u^\mu\right) u^\mu = -\left(\eta^{\mu\nu} + u^\mu u^\nu\right)\partial_\mu p_0 \text{ (eq. motion).}$$

Non-relativistic limit: $u^\mu \approx \left[1, v_x, v_y, v_z\right]$ & $p \ll \rho$.

Continuity: $\partial_\mu\left(\rho u^\mu\right) = 0 \Rightarrow \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right)$; motion: $\rho\left(\partial_\mu u^t\right) u^\mu + \eta^{t\mu}\partial_\mu p = 0 \Rightarrow \rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} = -\nabla p.$

GR units: [energy density] = kg-m^2/m^2/m^3 $\rightarrow$ kg/m^3; pressure = kg-m/ m^2/m^2 $\rightarrow$ kg/ m^3.



The sections of the above table in blue and green were not original to Einstein. It is from the Cauchy-stress tensor in continuum mechanics(fluid mechanics). What Einstein did was treat the time component(given as 0 superscript) as equivalent to the x, y and z components(given as 1, 2, and 3 superscripts) setting the stage for the above modified tensor. We will go through and analyze which have been tested and verified as contributing to gravitational fields and which have not.

Energy Density - This actually includes two types of particles:

Massive and Massless

Massive particles - have been thoroughly tested as they are the most prevalent form of energy where we are located. In fact, so much so that there would be no reason to cite any examples as **nearly every test** of General Relativity has related to massive particles.

Massless particles - such as gamma rays, photons, light, etc. ***have not been tested***. That is because their theorized contribution is usually quite insignificant.

The other dimensions of the above matrix are also difficult to test and detect:

Pressure, Shear Stress, and Momentum Density have not been tested, yet, as far as I know. There have been proposed tests, however: https://www.researchgate.net/pub...

Now, one could just as easily modify the source of gravity to be from energy which follows a geodesic only, such as mass, stress, and pressure(as opposed to a null geodesic, such as light). This would produce indistinguishable results because the energy contribution of non-mass T_{00}T00 of the tensor is considerably negligible(c^{-2}c−2). So there are some fine points which have not been conclusively settled by experiment and may open up new and interesting physics.
Or it may turn out massless particles and other non-massive forms of energy contribute, as well. Only time and better experiments will tell. Thank you for reading!

Electromagnetic Stress-Energy Tensor

$$T^{\mu\nu} = \begin{bmatrix} \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) & \frac{S_x}{c} & \frac{S_y}{c} & \frac{S_z}{c} \\ \frac{S_x}{c} & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{S_y}{c} & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{S_z}{c} & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix}$$

where $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B} =$ Poynting vector and $\sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\delta_{ij}$.

Gravitational Waves

Graviational waves far from their source will be very small perturbations of flat spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \text{ where } h_{\mu\nu} = h_{\nu\mu} \text{ and } \left|h_{\mu\nu}\right| \ll 1.$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \text{ where } h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}.$$

Riemann tensor: $R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}\right)$ to 1st order in $h_{\mu\nu}$.

Einstein eq.:

$$\boxed{\begin{gathered} G^{\gamma\sigma} \equiv R^{\gamma\sigma} - \frac{1}{2}g^{\gamma\sigma}R = 8\pi G T^{\gamma\sigma} = \\ \frac{1}{2}\left(\partial^\gamma\partial_\mu h^{\mu\sigma} + \partial^\sigma\partial_\mu h^{\mu\gamma} - \partial^\gamma\partial^\sigma h - \partial^\mu\partial_\mu h^{\gamma\sigma} - \eta^{\lambda\sigma}\partial_\beta\partial_\mu h^{\mu\beta} + \eta^{\lambda\sigma}\partial^\mu\partial_\mu h\right) \text{ where } \boxed{h = \eta^{\mu\nu}h_{\mu\nu}}. \end{gathered}}$$

Define: $\boxed{H_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h}$; define $\boxed{H \equiv \eta^{\eta\nu}H_{\mu\nu}} = \eta^{\eta\nu}\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = \left(1 - \frac{1}{2}4\right)h = -h; \therefore h_{\mu\nu} = H_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}H.$

Einstein eq.:

$$\boxed{\Box^2 H^{\gamma\sigma} - \partial^\gamma\partial_\mu H^{\mu\sigma} - \partial^\alpha\partial_\mu H^{\mu\gamma} + \eta^{\gamma\sigma}\partial_\beta\partial_\mu H^{\mu\beta} = -16\pi G T^{\gamma\sigma}} \text{ where } \boxed{\Box^2 \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \partial^\beta\partial_\beta = -\frac{\partial^2}{\partial t^2} + \nabla^2}.$$

Apply a gauge transformation such that

$$\boxed{\partial_\mu H'^{\mu\nu} = 0 = \text{Lorentz gauge}}; \text{ then Einstein eq.: } \boxed{\Box^2 H^{\mu\nu} = -16\pi G T^{\mu\nu} \ \&\ \partial_\mu H^{\mu\nu} = 0}.$$

Try a **plane-wave solution**:

$$\boxed{\begin{gathered} H^{\mu\nu}(t,z,y,z) = A^{\mu\nu}\cos\left(k_\sigma x^\sigma\right) = A^{\mu\nu}\cos\left(\vec{k}\cdot\vec{r} - \omega t\right) \text{ where } A^{\mu\nu} = \text{constant} \ \&\ k_\sigma = \left(-\omega, k_x, k_y, k_z\right). \\ \text{Wave speed} = v = \omega / k. \end{gathered}}$$

Requirements:

$$\text{Einstein eq.: } k^\alpha k_\alpha = 0 = k_\alpha k_\beta \eta^{\alpha\beta} = (\omega)^2\eta^{tt} + k_x^2\eta^{xx} + k_y^2\eta^{yy} + k_z^2\eta^{zz} = -\omega^2 + k^2 \Rightarrow \omega = k \ \&\ \boxed{v = \omega / k = 1}.$$

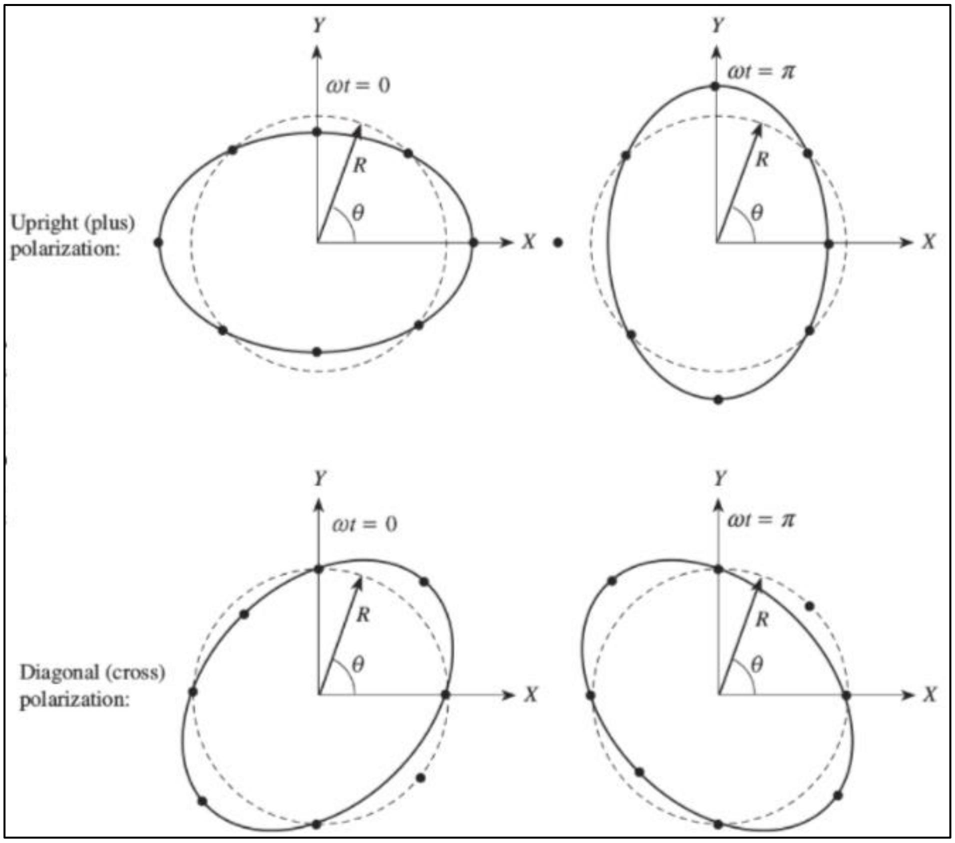
$$\text{Lorentz gauge: } k_\mu A^{\mu\nu} = 0 \ \&\ \text{Symmetry: } A^{\mu\nu} = A^{\nu\mu}.$$

$$\text{For a transverse-traceless (TT) gauge (comoving) transformation: } A^{t\mu} = A^{\mu t} \ \&\ A^\mu_\mu = 0.$$

$$\text{Propagating in the z direction: } k_\sigma = \left(-\omega, 0, 0, \omega\right).$$

$$\text{Only } A^{xy} = A^{yx} \equiv A_\times \ \&\ A^{xx} = -A^{yy} \equiv A_+ \text{ are nonzero.}$$

$$A^{\mu\nu} = A_+\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + A_\times\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}: \text{ 2 polarizations } + \text{ (upright) } \&\ \times \text{ (diagonal).}$$



Physical separation of one of the particles from the circle center is:

$$\text{Upright wave } (+)\text{: } \Delta s = R\left(1 + A_{+} \cos \omega t \cos 2\theta\right)$$
$$\text{Diagonal wave } (\times)\text{: } \Delta s = R\left(1 + A_{\times} \cos \omega t \sin 2\theta\right)$$

Typical astrophysical sources produce GW with amplitude ~10^{-20} measured at the Earth. Free particles separated by 1000 km will oscillate with amplitude of ~10^{-14} m, about the size of a large atomic nucleus. Although a 1-kHz wave has an intensity of ~30 watts/m^2, the interaction is weak. If the Sun were swallowed by a solar-mass black hole the GW amplitude would be ~10^{-8}, not noticeable by your body.

Gravitational-Wave Energy

The stress-energy tensor $T^{\mu\nu}$ describe the density of matter and energy excluding energy of the gravitational field, which is embedded in the Einstein tensor $G^{\mu\nu}$.

Gravitomagnetism in Weak-Field Limit

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \text{ where } h_{\mu\nu} = h_{\nu\mu} \;\&\; \left|h_{\mu\nu}\right| \ll 1;\text{ Lorentz gauge: } 0 = \eta^{\alpha\mu}\left(\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial_\nu h_{\alpha\mu}\right).$$

$$\text{Einstein Eq.: } \partial^\alpha \partial_\alpha h^{\mu\nu} = \left(\partial_t + \nabla^2\right) h^{\mu\nu} = -16\pi G\left(T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} T\right).$$

The solution is $h^{\mu\nu}\left(t,\vec{R}\right) = 4G\int_{src}\frac{1}{s}\left[T^{\mu\nu}\left(t-s,\vec{r}\right) - \frac{1}{2}\eta^{\mu\nu}T\left(t-s,\vec{r}\right)\right]dV$ where $\boxed{s_i \equiv \left|\vec{R}-\vec{r}_i\right|}$.

Assume slow-source $\left[\begin{array}{l}\left(p_0 = 0, u^t = 1, u^i \approx v^i\right); T^{\mu\nu} = \left(\rho_0 + p_0\right)u^\mu u^\nu + p_0 g^{\mu\nu} \Rightarrow T^{tt} \approx \rho_0, T^{ti} \approx \rho_0 v^i, T^{ij} \approx 0. \\ T = -\rho_0, T^{tt} - \frac{1}{2}\eta^{tt}T = \frac{1}{2}\rho_0, T^{ti} - \frac{1}{2}\eta^{ti}T \approx \rho_0 v^i, T^{ij} - \frac{1}{2}\eta^{ij}T \approx \frac{1}{2}\eta^{ij}\rho_0.\end{array}\right.$

Then $h^{tt}\left(t,\vec{R}\right) = h^{xx} = h^{yy} = h^{yy} = 2\int_{src}\frac{1}{s}G\rho_0\left(t-s,\vec{r}\right)dV \;\&\; h^{ti}\left(t,\vec{R}\right) = h^{it} = 4\int_{src}\frac{1}{s}GJ^i\left(t-s,\vec{r}\right)dV$ where $\boxed{\vec{J} \equiv \rho_0\vec{v}}$.

Define gravitational scalar & vector potentials: $\boxed{\Phi_G \equiv -\frac{1}{2}h^{tt} = -\int_{src}\frac{G\rho_0}{s}dV}$ & $\boxed{A_G^i \equiv -\frac{1}{4}h^{ti} = -\frac{1}{4}h^{it} = -\int_{src}\frac{GJ^i}{s}dV}$.

Gravitational Maxwell Equations: $\boxed{\vec{\nabla}\cdot\vec{A}_G = -\frac{\partial\Phi_G}{\partial t}}$, $\boxed{\vec{E}_G \equiv \vec{\nabla}\Phi_G - \frac{\partial\vec{A}_G}{\partial t}}$, $\boxed{\vec{B}_G = \vec{\nabla}\times\vec{A}_G}$.

Gravitoelectric & gravitomagnetic fields:

$$\boxed{\vec{\nabla}\cdot\vec{E}_G = -4\pi G\rho_0}, \boxed{\vec{\nabla}\times\vec{B}_G - \frac{\partial\vec{E}_G}{\partial t} = -4\pi G\vec{J}}, \boxed{\vec{\nabla}\cdot\vec{B}_G = 0}, \boxed{\vec{\nabla}\times\vec{E}_G + \frac{\partial\vec{B}_G}{\partial t} = 0}.$$

The minus signs in the first two equations are because the gravitational force is always attractive.

The geodesic equation for low speeds is

$$\frac{d^2x^i}{dt^2} \approx \frac{1}{2}\eta^{ik}\partial_k h_{tt} + \eta^{ik}\left(\partial_k h_{tj} - \partial_j h_{tk}\right)V^j \Rightarrow \boxed{\vec{F}_G = m\frac{d^2\vec{x}}{dt^2} = m\left(\vec{E}_G + 4\vec{V}\times\vec{B}_g\right)}.$$

The gravitomagnetic force is 4 times the electromagnetic force and the sign of $\vec{B}_G$ is reversed $\Rightarrow$ a left-hand rule.

Gravitomagnetic Effects on a Gyroscope. We know from electromagnetic theory that a simple current loop of area A carrying current i has a magnetic moment $\vec{\mu}$ with magnitude $\mu = iA$ pointing perpendicular to the loop's plane in the sense indicated by your right thumb when your right fingers curl in the direction of the current flow. In a magnetic field $\vec{B}$, such a current loop experiences a torque $\vec{\tau} = \vec{\mu} \times \vec{B}$ that seeks to align the loop's magnetic moment with the field.

You can show (see box 35.4) that a spinning object (a gyroscope) has an analogous gravitomagnetic moment $\vec{\mu}_G = \frac{1}{2}\vec{s}$, where $\vec{s}$ is the gyroscope's total spin angular momentum. By analogy, in a gravitomagnetic field $\vec{B}_G$, such a gyroscope should experience a torque

$$\vec{\tau} = \vec{\mu}_G \times 4\vec{B}_G = \vec{s} \times 2\vec{B}_G \tag{35.13}$$

(remember that a gravitomagnetic field exerts 4 times more force on a moving mass than the corresponding magnetic field would exert on a moving charge). As discussed in box 35.5, exerting such a torque on a gyroscope causes it to precess around the field direction with an angular velocity of

$$\vec{\Omega}_{LT} = -2\vec{B}_G \tag{35.14}$$

if we define $\vec{\Omega}_{LT}$ to point as your right thumb does when your fingers curl in the direction of the precession. Observing this so-called **Lense-Thirring precession** of a gyroscope at a point in empty space provides a practical way to measure both the magnitude and direction of any gravitomagnetic field present at that location.

Lense-Thirring Precession Near a Spinning Object. Another established result from electromagnetic theory is that any steadily spinning charged object with spherical symmetry produces a dipole magnetic field in its exterior:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi r^3}[3(\vec{\mu}\cdot\hat{r})\hat{r} - \vec{\mu}] \tag{35.15}$$

where $\vec{\mu}$ is the object's total magnetic moment, $\mu_0 = 4\pi k$ (in GR units), $\vec{r}$ is the displacement from the object's center to the point where the field is being evaluated, and $\hat{r}$ is a unit vector pointing in the $\vec{r}$ direction (see the "Dipole" entry in Wikipedia). By analogy, then, the gravitomagnetic field produced by a spherical star or planet with total spin angular momentum $\vec{S}$ is

$$\vec{B}_G(\vec{r}) = -\frac{G}{r^3}[3(\vec{\mu}_G\cdot\hat{r})\hat{r} - \vec{\mu}_G] = \frac{G}{2r^3}[\vec{S} - 3(\vec{S}\cdot\hat{r})\hat{r}] \tag{35.16}$$

where the factor of 2 comes from $\vec{\mu}_G = \frac{1}{2}\vec{S}$ (as discussed above) and the minus sign comes from the reversal of the gravitomagnetic field compared to the analogous magnetic field). We can use this to estimate both the magnitude and direction of the Lense-Thirring effect near a rotating body of interest.

For example, consider a gyroscope in an equatorial orbit around the earth. Since the earth's spin angular momentum $\vec{S}$ points perpendicular to the earth's equatorial plane from south to north (check with your right hand), for a point on the earth's equatorial plane, $\vec{S}\cdot\hat{r} = 0$, so on that plane, $\vec{B}_G = G\vec{S}/2r^3$ oriented parallel to the earth's spin $\vec{S}$. The orbiting gyroscope's precession relative to distant stars will be easiest to observe if its spin $\vec{s}$ is perpendicular to this direction (i.e, it lies *in* the equatorial plane): let's assume this. The angular speed of precession will then be

$$\Omega_{LT} = 2B_G = \frac{GS}{r^3} = \frac{GI\omega}{r^3} \tag{35.17}$$

where I is the earth's moment of inertia and ω is its spin angular speed = 2π/day.

To go further, we need to estimate the earth's moment of inertia. We can quite generally express an axially symmetric object's moment of inertia as

$$I = \alpha MR^2 \tag{35.18}$$

where M is the object's mass, R is its radius, and α is a constant ($0 < \alpha \leq 1$) that depends on the distribution of mass in the object: α larger if the mass is concentrated near the object's rim and smaller if it is concentrated in the center. For a uniform sphere, $\alpha = 2/5$, but since the earth is denser near its center, we would expect α for the earth to be somewhat smaller: detailed estimates based on the earth's measured density profile imply that $\alpha = 0.33$. Therefore, a good estimate of the Lense-Thirring precession rate for our orbiting gyroscope would be

$$\Omega_{LT} = \frac{G(\alpha MR^2)\omega}{r^3} = 0.33\frac{GM}{R}\left(\frac{R}{r}\right)^3 \omega \tag{35.19}$$

where R is the earth's radius $\approx$ 6380 km. Note that GM for the earth is 4.45 mm. Therefore, for a gyroscope in low-earth orbit where $r \approx R$, we have

$$\Omega_{LT} \approx \frac{0.33(4.5 \times 10^{-3}\ \cancel{\text{m}})}{6,380,000\ \cancel{\text{m}}}\left(\frac{2\pi\ \text{rad}}{\cancel{\text{day}}}\right)\left(\frac{365\ \cancel{\text{day}}}{\text{y}}\right) = \frac{5.4 \times 10^{-7}\ \text{rad}}{\text{y}} \tag{35.20}$$

This corresponds to about 0.11 arcseconds per year, which is obviously a very small number (and therefore is very difficult to measure).

Geodetic Precession. There is a second effect that will also cause our hypothetical orbiting gyroscope to precess. Because of the curvature of spacetime, a gyroscope orbiting even a non-spinning object will precess: this phenomenon is called *geodetic precession*. Since this is not a gravitomagnetic effect, I will not discuss it here, but problem P35.7 will guide you through the derivation if you are interested. Again assuming that the gyroscope orbits in the equatorial plane and has its spin lying in that plane, the angle through which the gyroscope precesses is

$$\Delta\phi_{gd} \approx \frac{3\pi GM}{R}\ \text{ per orbit} \tag{35.21}$$

Since a near-earth orbit takes about 85 minutes, the precession rate is

$$\begin{aligned}\Omega_{gd} = \frac{\Delta\phi_{gd}}{T} &= \frac{3\pi(4.45 \times 10^{-3}\ \cancel{\text{m}})}{(85\ \cancel{\text{min}})(6,380,000\ \cancel{\text{m}})}\left(\frac{1\ \cancel{\text{min}}}{60\ \cancel{\text{s}}}\right)\left(\frac{3.16 \times 10^{7}\ \cancel{\text{s}}}{1\ \text{y}}\right) \\ &= 4.1 \times 10^{-5}\ \text{rad/y}\end{aligned} \tag{35.22}$$

This is almost two orders of magnitude larger than the Lense-Thirring effect for a gyroscope in low earth orbit.

Carter Constant

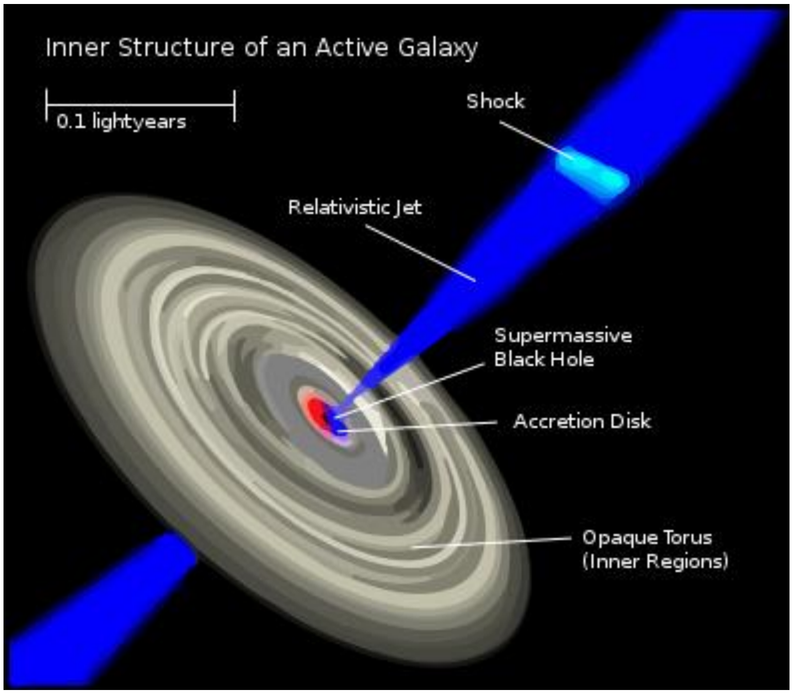
$C \equiv p_\theta^2 + \cos^2\theta\left(a^2\left(1-e^2\right)+\frac{\ell_z^2}{\sin^2\theta}\right) =$ Carter Constant, which is a function of three conserved quantities

p_θ, e and ℓ_z. For the Schwarzschild metric $(a=0)$ $\boxed{C \equiv p_\theta + \frac{\ell_z^2}{\sin^2\theta}}$. For equatorial motion: $(\theta = \pi/2)$ $\boxed{C \equiv \ell_z^2}$.

Some authors use the symbol Q instead of C for the Carter Constant.

Define $\boxed{L = \sqrt{\ell_z^2 + Q}} =$ total angular momentum for the Schwarzschild metric $(a=0)$. That is, for $a = 0: Q = \ell_x^2 + \ell_y^2$.

Black Hole Relativistic Jets



"They likely arise from dynamic interactions within accretion disks, whose active processes are commonly connected with compact central objects such as black holes, neutron stars or pulsars. One explanation is that tangled magnetic fields[2] are organized to aim two diametrically opposing beams away from the central source by angles only several degrees wide. (c.>1%.).[3] Jets may also be influenced by a general relativity effect known as frame-dragging."

"Because of the enormous amount of energy needed to launch a relativistic jet, some jets are possibly powered by spinning black holes. However, the frequency of high-energy astrophysical sources with jets suggest combination of different mechanisms indirectly identified with the energy within the associated accretion disk and X-ray emissions from the generating source. Two early theories have been used to explain how energy can be transferred from a black hole into an astrophysical jet:

- *Blandford–Znajek process*.[13] This theory explains the extraction of energy from magnetic fields around an accretion disk, which are dragged and twisted by the spin of the black hole. Relativistic material is then feasibly launched by the tightening of the field lines.
- *Penrose mechanism*.[14] Here energy is extracted from a rotating black hole by frame dragging, which was later theoretically proven to be able to extract relativistic particle energy and momentum, and subsequently shown to be a possible mechanism for jet formation."

“Jets may also be observed from spinning neutron stars. An example is pulsar IGR J11014-6103, which has the largest jet so far observed in the Milky Way Galaxy whose velocity is estimated at 80% the speed of light. (0.8c.) X-ray observations have been obtained but there is no detected radio signature or accretion disk. Initially, this pulsar was presumed to be rapidly spinning but later measurements indicate the spin rate is only 15.9 Hz.[19][20] Such a slow spin rate and lack of accretion material suggest the jet is neither rotation nor accretion powered, though it appears aligned with the pulsar rotation axis and perpendicular to the pulsar's true motion.”

References

- General Relativity by Christopher M Hirata
- Lorentz Covariance
- General Covariance
- Accelerating Universe from a Newtonian View
- Accelerating Universe and Galaxy Stars’ Velocity using a Revised Newton Law of Gravity
- Modified Newtonian Dynamics (MOND)
- General Relativity Workbook
- GRW Sample Chapters
- https://www.redshelf.com/book/515/a-general-relativity-workbook-515-9781938787324-thomas-a-moore
- http://pages.pomona.edu/~tmoore/grw/Resources/GRWBook.pdf
- https://warwick.ac.uk/fac/sci/physics/current/teach/module_home/px436/notes/lecture15.pdf
- http://www.physics.usu.edu/Wheeler/GenRel/Lectures/GRNotesDecSchwarzschildGeodesicsPost.pdf
- http://physicspages.com/pdf/Moore/Moore%20Problems%2023.03.pdf
- https://en.wikipedia.org/wiki/Proper_time
- http://hyperphysics.phy-astr.gsu.edu/hbase/Relativ/tdil.html
- http://physicspages.com/Moore%20Relativity.html
- https://en.wikipedia.org/wiki/Rindler_coordinates
- https://en.wikipedia.org/wiki/Hawking_radiation#cite_note-kumar2012-12
- The Kerr Spacetime: A Brief Introduction
- Lecture Notes on General Relativity
- Formulas in Riemannian Geometry
- https://en.wikipedia.org/wiki/Numerical_relativity
- https://www.seas.upenn.edu/~amyers/NaturalUnits.pdf
- https://en.wikipedia.org/wiki/Natural_units#Planck_units
- https://medium.com/starts-with-a-bang/how-do-black-holes-evaporate-5463dbda6832
- http://eagle.phys.utk.edu/guidry/astro616/lectures/lecture_ch16.pdf General Covariance
- http://www.blau.itp.unibe.ch/newlecturesGR.pdf
- https://jila.colorado.edu/~ajsh/astr3740_17/grbook.pdf

Appendix: SI & GR Units

- c = 299,792,458 m/s
- 1 m = 1/299,792,458 s = 3.3564095 x 10^{-9} s = 3.34 ns
- 1 μs = 299.8 m (of time); 1 ms = 299,800 km; 1 s = 299.800 km; 1 min = 17.99 x 10^6 km
- 1 hr = 1.079 x 10^9 km; 1 day = 25.90 x 10^9 km; 1 yr = 9.461 x ao^15 m
- universe age = 13.7 Gyr = 1.30 x 10^26 m

We can convert any quantity in SI units into GR units by multiplying by as many factors of the conversion factor $c = 1 = (2.99792458 \times 10^8 \text{ m} / 1 \text{ s})$ as are required to eliminate all units of seconds from the quantity.

$$1\,\text{J} = 1\,\text{kg}\frac{\cancel{\text{m}}^2}{\cancel{\text{s}}^2}\left(\frac{1\,\cancel{\text{s}}}{299{,}792{,}458\,\cancel{\text{m}}}\right)^2 = 1.1126501 \times 10^{-17}\,\text{kg (energy)} \quad (2.9a)$$

$$1\text{ kg (energy)} = 8.98755179 \times 10^{16}\text{ J} \quad (2.9b)$$

$$1\text{ kg (momentum)} = 299{,}792{,}458\text{ kg}\cdot\text{m/s} \quad (2.9c)$$

$$1\text{ eV} = 1.602 \times 10^{-19}\text{ J} = 1.782 \times 10^{-36}\text{ kg (energy)} \quad (2.9d)$$

$$1\text{ eV (momentum)} = 5.34 \times 10^{-28}\text{ kg}\cdot\text{m/s} \quad (2.9e)$$

The eV as a unit for mass, momentum, and energy has a much more convenient size than the kilogram when dealing with subatomic particles, atoms, and molecules.

Here are some constants in GR units that will be useful to us later:

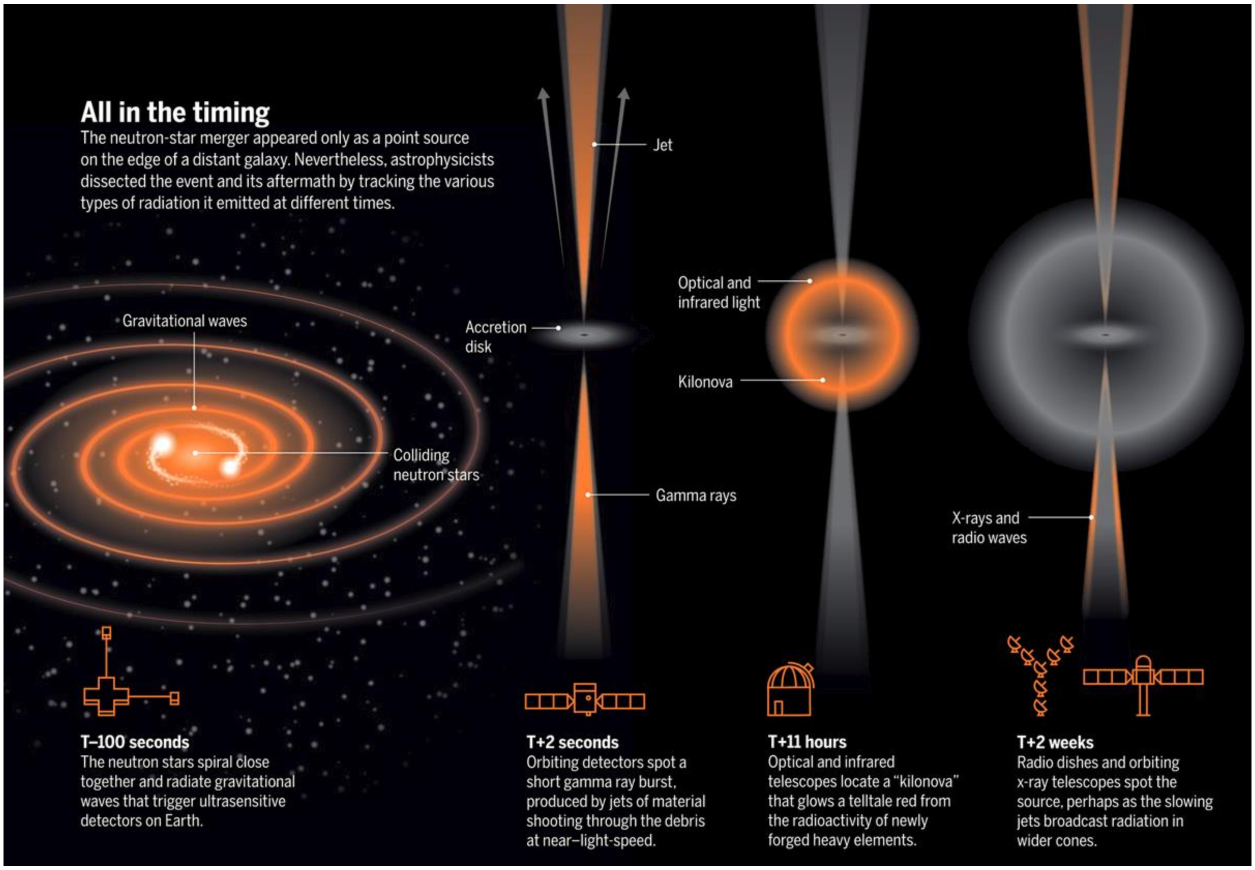
$$g = 1.09 \times 10^{-16}\text{ m}^{-1} = 1/(9.17 \times 10^{15}\text{ m}) \approx 1/(1\text{ ly}) \quad (2.10a)$$

$$G = 7.426 \times 10^{-28}\text{ m/kg} = 1477\text{ m / (solar mass)} \quad (2.10b)$$

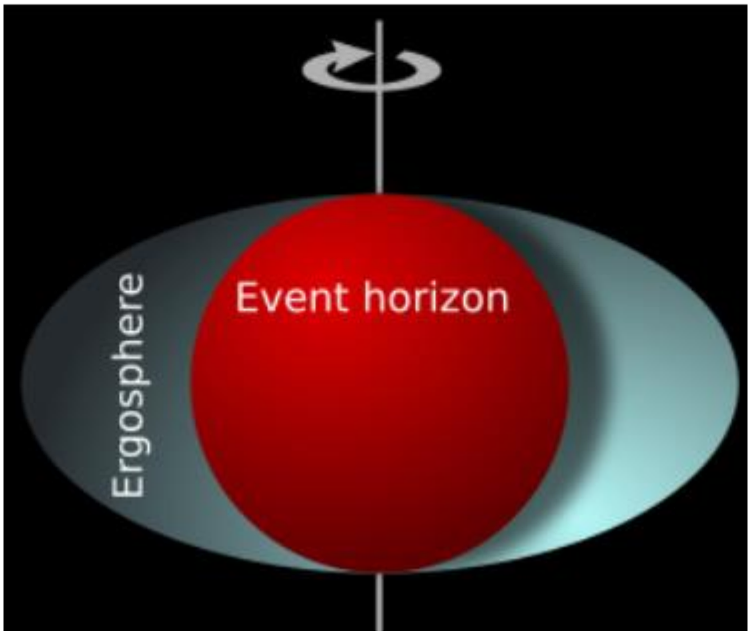
Planck mass: $\boxed{m_p \equiv \sqrt{\frac{\hbar c}{G}}} = (2.176470 \pm 51) \times 10^{-8}\text{ kg} = 1.220910 \times 10^{19}\text{ GeV/c}^2$

Planck length: $\boxed{\ell_p = \sqrt{\frac{\hbar G}{c^3}}} = (1.616229 \pm 38) \times 10^{-35}\text{m}$

Planck time: $\boxed{t_p = \sqrt{\frac{\hbar G}{c^5}}} = 5.3912 \times 10^{-44} s$



http://www.sciencemagazinedigital.org/sciencemagazine/22_december_2017?sub_id=d4lGvcb0nqxn&u1=41644052&pg=32#pg32



http://chartasg.people.cofc.edu/chartas/Teaching_files/phys412_ch4.pdf