

Positive-Feedback Mathematics

L. David Roper

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There are many physical situations where two or more variables mutually interact to create positive feedback. An example is Earth average temperature and carbon-dioxide concentration in the atmosphere.

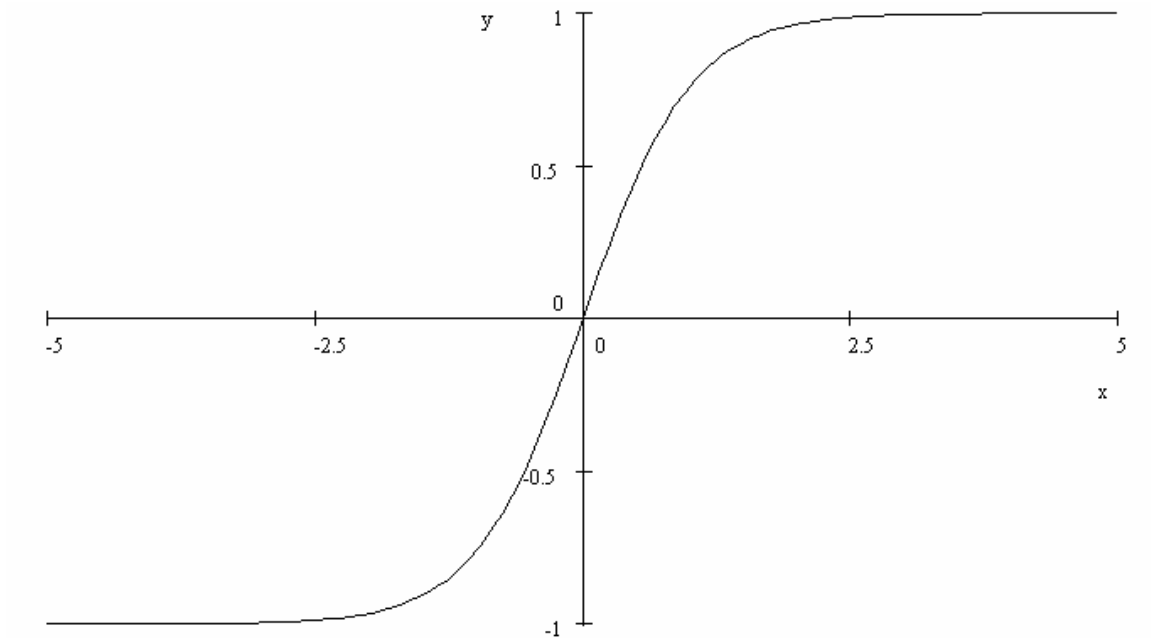
This article devises a simple model to account for positive feedback between two variables.

Consider two variables, A and B, that mutually interact to create positive feedback. Assume that the feedback time dependence is governed by the hyperbolic tangent function, $\tanh(x)$. (See <http://www.roperld.com/science/hyperbolictangentworld.pdf>). That is, there is a time delay for one variable to react to the change in the other variable in the same direction; the time delay may be different for each of the two variables.

The following form will be used to incorporate the hyperbolic tangent function in the feedback mechanism:

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \left\{ \begin{array}{l} \xrightarrow{x \rightarrow -\infty} -1 \\ \xrightarrow{0 > x \gg -1} e^{2x} - 1 \\ \xrightarrow{x \rightarrow 0} 0 \\ \xrightarrow{0 < x \ll +1} e^{2x} - 1 \\ \xrightarrow{x \rightarrow +\infty} +1 \end{array} \right. .$$

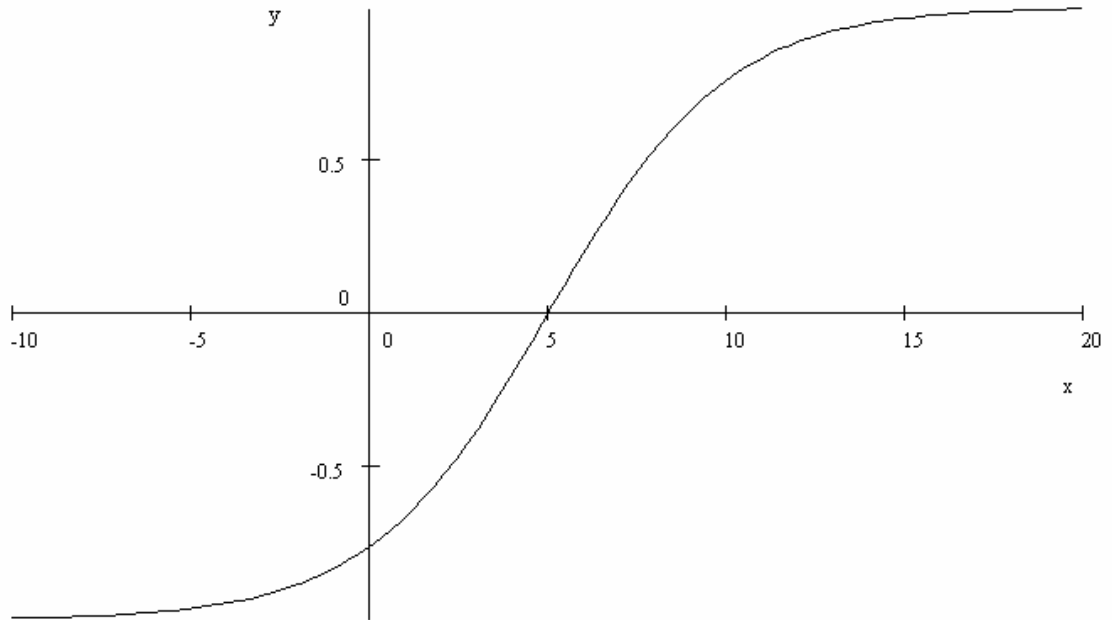
Its graph is as follows:



A more general function is needed for real-World fits to natural phenomena, as given in the following equation.

$$\tanh\left(\frac{t-t_m}{2w}\right)$$

For example, $\tanh\left(\frac{t-5}{5}\right)$ is:



The approach for feedback among two variables will be an iterative one that converges very rapidly.

For the simplest case, consider the case that variable A changes abruptly from some steady-state value to a new value by an increment ΔA . Then assume that, to a first order approximation, variable B changes gradually by the hyperbolic-tangent equation with the mid-point at the time of the abrupt change of A to some final equilibrium change in value b . This is a reasonable behavior to expect for the variable B when variable A changes abruptly. That is, to a first-order approximation:

$$dA = \Delta A,$$

$$dB = b \tanh\left(\frac{t-t_m}{2w_B}\right)\Delta A = \Delta A \Gamma_b,$$

where t_m = time of abrupt change of A

$$\text{and } \Gamma_b \equiv b \tanh\left(\frac{t-t_m}{2w_m}\right).$$

Similarly, when B changes as above in first order, A then changes gradually by the hyperbolic-tangent equation with the mid-point at the time of the original abrupt time of change of A to some final equilibrium change in value a .

So, the second-order approximation is:

$$dA = \Delta A + a \tanh\left(\frac{t-t_m}{2w_A}\right) dB = \Delta A + \Gamma_A dB = \Delta A(1 + \Gamma_A \Gamma_B) \text{ and}$$

$$dB = \Gamma_B dA = \Delta A \Gamma_B (1 + \Gamma_A \Gamma_B), \text{ where}$$

$$\Gamma_A \equiv a \tanh\left(\frac{t-t_m}{2w_A}\right).$$

The third-order approximation is:

$$dA = \Delta A + \Gamma_A dB = \Delta A \left[1 + \Gamma_A \Gamma_B + (\Gamma_A \Gamma_B)^2 \right],$$

$$dB = \Gamma_B dA = \Delta A \Gamma_B \left[1 + \Gamma_A \Gamma_B + (\Gamma_A \Gamma_B)^2 \right].$$

The fourth-order approximation is:

$$dA = \Delta A + \Gamma_A dB = \Delta A \left[1 + \Gamma_A \Gamma_B + (\Gamma_A \Gamma_B)^2 + (\Gamma_A \Gamma_B)^3 \right],$$

$$dB = \Gamma_B dA = \Delta A \Gamma_B \left[1 + \Gamma_A \Gamma_B + (\Gamma_A \Gamma_B)^2 + (\Gamma_A \Gamma_B)^3 \right].$$

The Nth-order approximation is:

$$dA = \Delta A \sum_{n=0}^N (\Gamma_A \Gamma_B)^n \text{ and}$$

$$dB = \Delta A \Gamma_B \sum_{n=0}^N (\Gamma_A \Gamma_B)^n.$$

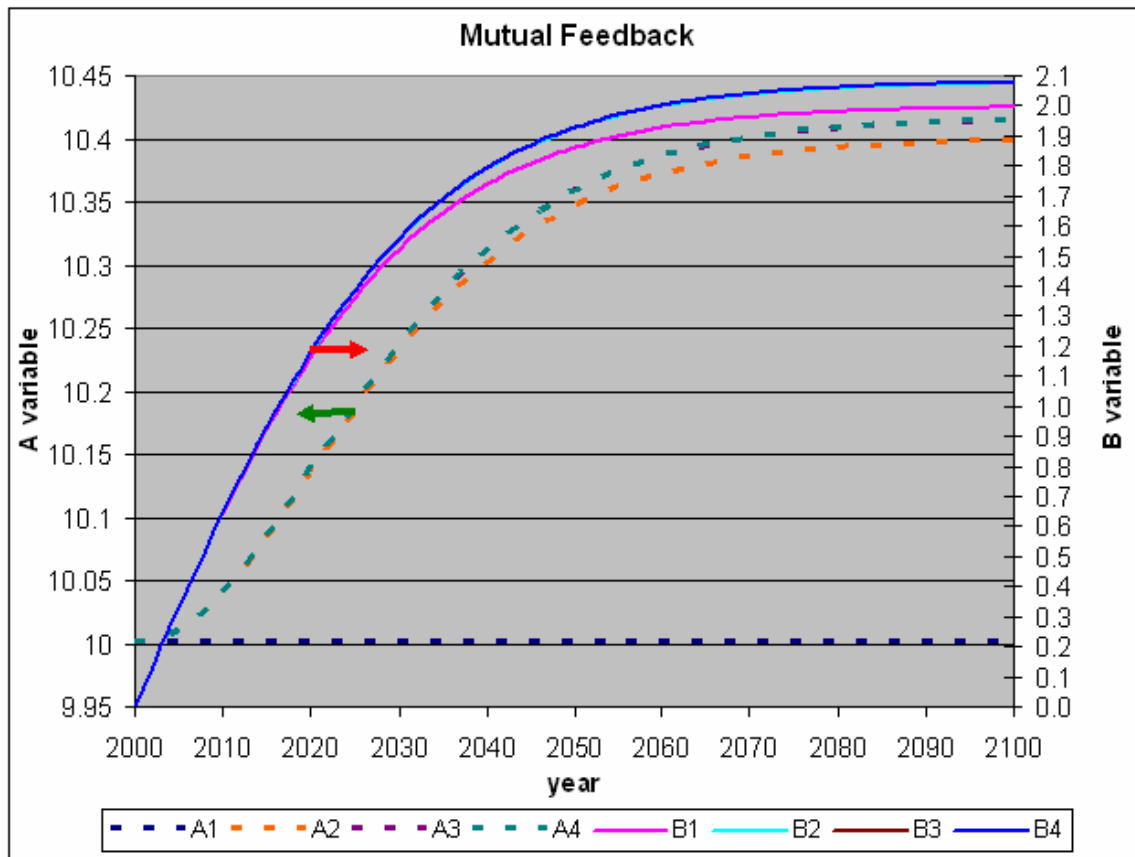
The total result would be when $N = \infty$. However, the series may converge for just a few terms because of the asymptotic behavior of the hyperbolic tangent.

Consider the simple case where the mutual interaction is symmetrical:

$$\Delta A = 10, a = b = 0.2, t_A = t_B = 2000 \text{ and } w_A = w_B = 15.$$

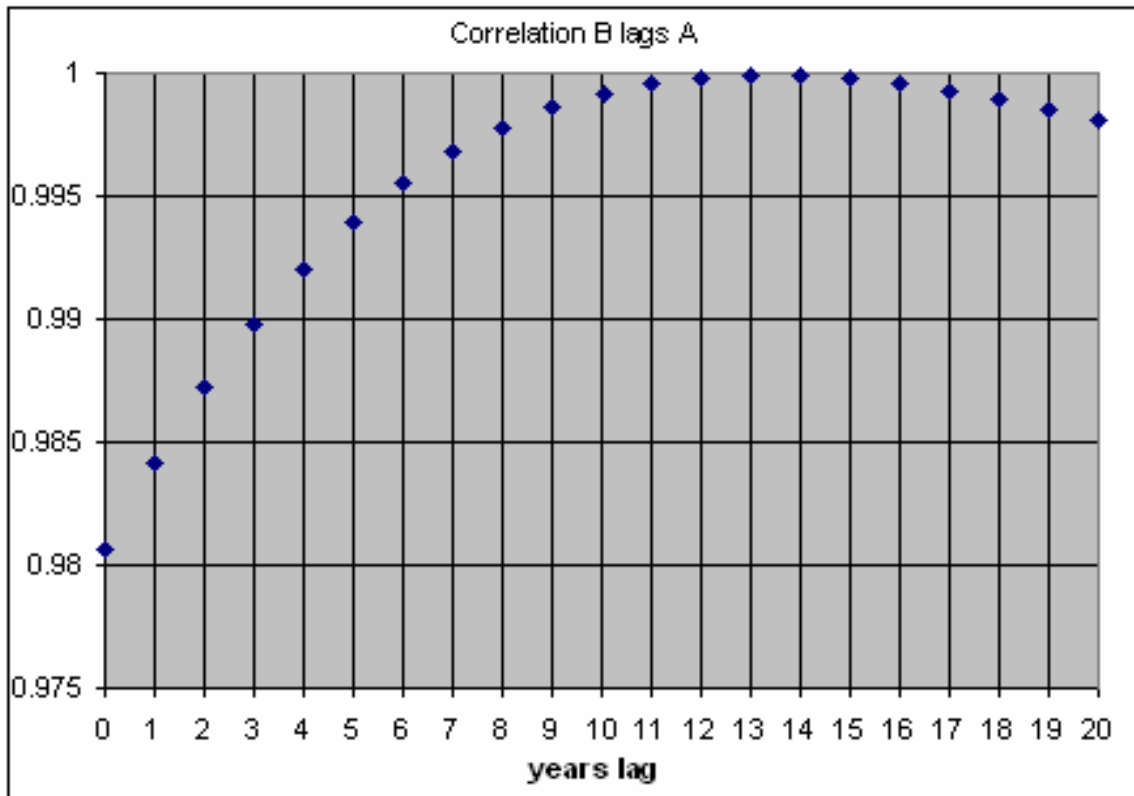
That is, the time delay is the same for B when A changes as it is for A when B changes (parameters w_A and w_B) and the amount of change is the same for both A and B (parameters a and b).

The first four approximations are shown in this graph:



For this symmetric model highly accurate convergence occurs at the 3rd approximation and the result is quite good in the 2nd approximation, especially for variable B. Even the 1st approximation is a good estimate for variable B.

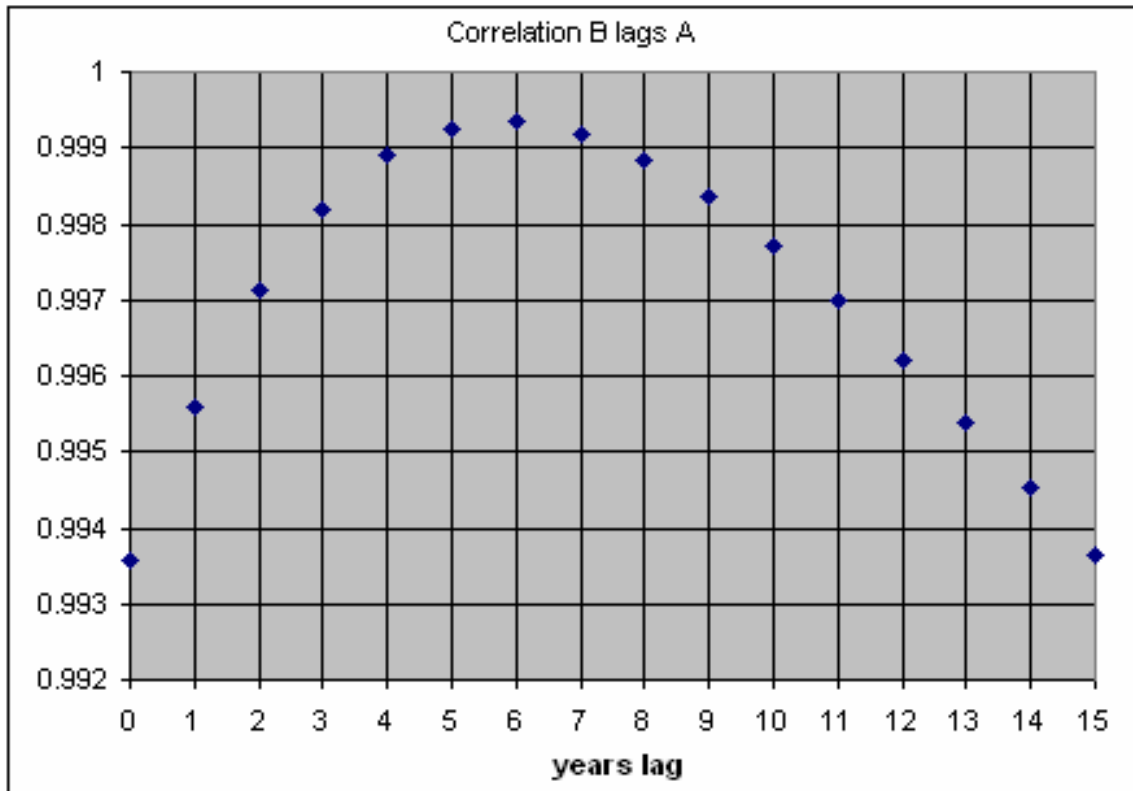
A correlation calculation for the 4th approximation gives the correlations:



So, B lags A by about 13.5 years, because A was changed first. If B were changed first, A would lag B by the same number of years for a symmetrical mutual interaction. Therefore, two variables that mutually interact symmetrically to give positive feedback will lead or lag each other by the same number of time units depending on which one is changed first after a steady state situation.

Consider correlations for the case where the mutual interaction is asymmetrical:

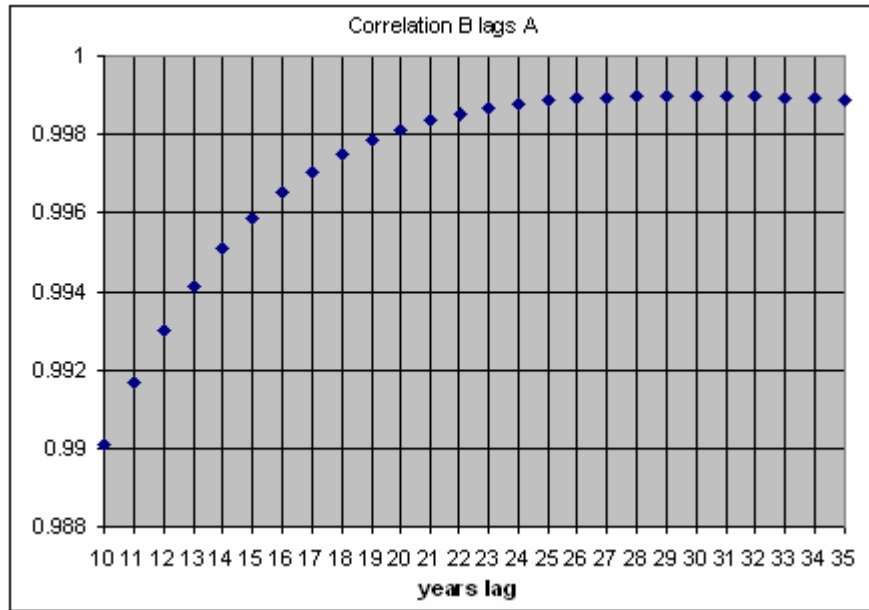
$$\Delta A = 10, a = 0.2, b = 0.1, t_A = t_B = 2000 \text{ and } w_A = 10, w_B = 15.$$



The lag time of B after A is about 6 years.

Consider correlations for the mirror image asymmetrical case to the one just above:

$$\Delta A = 10, a = 0.1, b = 0.2, t_A = t_B = 2000 \text{ and } w_A = 15, w_B = 10.$$



The lag time of B after A is about 29 years.

It appears to be the case that B always lags A if A is given an abrupt change for this asymmetric model. So, as to which one of the variables leads depends on which one is changed abruptly from a steady state.